ON NOVIKOV-TYPE CONJECTURES

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This paper is based on lectures given by the second author at the 2000 Summer conference at Mt. Holyoke. We also added a brief epilogue, essentially "What there wasn't time for." Although the focus of the conference was on noncommutative geometry, the topic discussed was conventional commutative motivations for the circle of ideas related to the Novikov and Baum-Connes conjectures. While the article is mainly expository, we present here a few new results (due to the two of us).

It is interesting to note that while the period from 80's through the mid-90's has shown a remarkable convergence between index theory and surgery theory (or more generally, the classification of manifolds) largely motivated by the Novikov conjecture, most recently, a number of divergences has arisen. Possibly, these subjects are now diverging, but it also seems plausible that we are only now close to discovering truly deep phenomena and that the difference between these subjects is just one of these. Our belief is that, even after decades of mining this vein, the gold is not yet all gone.

As the reader might guess from the title, the focus of these notes is not quite on the Novikov conjecture itself, but rather on a collection of problems that are suggested by heuristics, analogies and careful consideration of consequences. Many of the related conjectures are false, or, as far as we know, not directly mathematically related to the original conjecture; this is a good thing: we learn about the subtleties of the original problem, the boundaries of the associated phenomenon, and get to learn about other realms of mathematics.

Lecture One: Topology and K-theory

1. For the topologist, the Novikov conjecture is deeply embedded in one of the central projects of his field, that of classifying manifolds within a homotopy type up to homeomorphism or diffeomorphism. To put matters in perspective, let us begin by reviewing some early observations regarding this problem.

   The first quite nontrivial point is that there are closed manifolds that are homotopy equivalent but not diffeomorphic (homeomorphism is much more difficult). It is quite easy to give examples which are manifolds with boundary: the punctured torus and the thrice punctured 2-sphere are homotopy equivalent but not diffeomorphic; their boundaries have different numbers of components.

   The first class of examples without boundary are the lens spaces: quotients of the sphere by finite cyclic groups of isometries of the round metric. To be concrete, let S^{2n-1} be the unit sphere in $\mathbb{C}^n$ with coordinates $(u_1, \ldots, u_n)$. For any n-tuple of primitive k-th roots of unity $e^{2\pi i a_r/k}$, one has a $\mathbb{Z}_k$ action by multiplying the r-th coordinate by the r-th root of unity. The quotient manifolds under these actions are homotopy equivalent (preserving the identification of fundamental group with $\mathbb{Z}_k$) iff the products of the rotation numbers $a_1 a_2 \cdots a_n$ are the same mod k. On the other hand, these manifolds are diffeomorphic iff they are isometric iff the sets of rotation numbers are the same (i.e. they agree after reordering). There are essentially two different proofs of this fact, both of which depend on the same sophisticated number-theoretic fact, the Franz independence lemma.

   The first proof, due to de Rham, uses Reidemeister torsion. Since the cellular chain complex of a lens space is acyclic when tensored with $\mathbb{Q}[x]$ for x a primitive k-th root of unity, one gets a based (by cells) acyclic complex $0 \to C_{2n-1} \to \cdots \to C_0 \to 0$, which gives us a well-defined nonzero determinant element in $\mathbb{Q}[x]$ (now called the associated element of K_1). This quantity is well-defined up to multiplication by a root of unity (and a sign). One now has to check that these actually determine the rotation numbers, a fact verified by Franz's lemma. See [Mi] and [Co]. The second proof comes much later and is due to Atiyah and Bott [AB]. It uses index-theoretic ideas critically, and implies more about the topology of lens spaces. We will return to it a bit later.

After de Rham's theorem, it was very natural to ask, following Hurewicz, whether all homotopy equivalent simply-connected manifolds are diffeomorphic. (It was not until Milnor's examples of exotic spheres that mathematicians really considered seriously the existence of different categories of manifolds.) However, very classical results can be used to disprove this claim as well. Consider a sphere bundle over the sphere S^4 where the fiber is quite high-dimensional. Since $\pi_3(O(n)) = \mathbb{Z}$ for large n, we can construct an infinite number of these bundles by explicit clutching operations; their total spaces are distinguished by p_1. On the other hand, if we could nullhomotop the clutching maps in $\pi_3(\text{Isometries}(S^{n+1}))$ pushed into $\pi_3(\text{Selfmaps}(S^{n+1}))$, we would show that the total space is homotopy equivalent to a product. A little thought shows that $\pi_3(\text{Selfmaps}(S^{n+1}))$ is the same as the third stable homotopy group of spheres, which is finite by Serre's thesis. Combining this information, one quickly concludes that there are infinitely many manifolds homotopy equivalent to $S^4 \times S^{n+1}$ for large n, distinguished by p_1.

Much of our picture of high-dimensional manifolds comes from filtering the various strands arising in the above examples, analyzing them separately, and recombining them.

2. Before considering the parts that are most directly connected to operator K-theory, it is worthwhile to discuss the connection between classification of manifolds and algebraic K-theory.

The aforementioned Reidemeister torsion invariant is an invariant of complexes defined under an acyclicity hypothesis. It is a computationally feasible shadow of a more basic invariant of homotopy equivalences, namely Whitehead torsion.

Let X and Y be finite complexes and $f : X \to Y$ a homotopy equivalence. Then using the chain complex of the mapping cylinder of f rel X or its universal cover, one obtains as before a finite-dimensional acyclic chain complex of based $\mathbb{Z}\pi$ chain complexes. The torsion $\tau(f)$ of f is the element of $K_1(\mathbb{Z}\pi)$ determined by means of the determinant, up to the indeterminacy of basis, which is a sign and element of π (viewed as a 1×1 matrix over the group ring). The quotient $K_1(\mathbb{Z}\pi)/\pm\pi$ is denoted by $\mathrm{Wh}(\pi)$.

A geometric interpretation of the vanishing of $\tau(f)$ is the following: say that X and Y are stably diffeomorphic (or, more naturally for this discussion, PL homeomorphic) if their regular neighborhoods in Euclidean space are diffeomorphic. The quantity $\tau(f)$ vanishes iff f is homotopic to a diffeomorphism between thickenings of X and Y. A homotopy equivalence with vanishing torsion is called a ***simple homotopy equivalence***. As before, we recommend [Co, Mi] for Whitehead's theory of simple homotopy and [RS] for the theory of regular neighborhoods.

Remark: If we require X and Y to be manifolds, then one can ask that the stabilization only allow taking products with disks. Doing such does change the notion: the entire difference however is that we have discarded the topological K-theory. Two manifolds will be stably diffeomorphic in this restricted sense iff they have the same stable tangent bundle (in KO, or KPL for the PL analogue) and are simple homotopy equivalent. The proof of this fact is no harder than the polyhedral result.

3. Much deeper are unstable results. The prime example is Smale's h-cobordism theorem (or the Barden-Mazur-Stallings extension thereof).

Theorem: Let M^n be a closed manifold of dimension at least 5; then $\{W^{n+1}\colon M$ is one of two components of the boundary of W, and W deform retracts to both$\}$/diffeomorphism (or PL homeomorphism or homeomorphism) is in $1-1$ correspondence with $\mathrm{Wh}(\pi)$.

The various W in the theorem are called h-cobordisms. The significance of this theorem should be obvious: it provides a way to produce diffeomorphisms from homotopy data. As such, it stands behind almost all of the high-dimensional classification theorems.

4. The proof that $\mathrm{Wh}(0) = 0$ is an easy argument using linear algebra and the Euclidean algorithm. Thus, in the simply-connected case the h-cobordisms are products. In particular, every homotopy sphere is the union of two balls and an h-cobordism that runs between their boundaries. The h-cobordism theorem asserts that the h-cobordism is just an annulus; since the union of a ball and an annulus is a ball, one can show that every homotopy sphere can be obtained by glueing two balls together along their boundary. This result implies the Poincaré conjecture in high dimensions: every homotopy sphere is a PL sphere. Using versions for manifolds with boundary, one can quickly prove the following theorems.

   Zeeman unknotting theorem: Every proper embedding[1] of D^n in D^{n+k} for $k > 2$ is PL or differentiably trivial.

   Rothenberg-Sondow Theorem: If p is a prime number, smooth $\mathbb{Z}_p$ actions on the disk whose fixed set is a disk of codimension exceeding 2 are determined by an element of $\mathrm{Wh}(\mathbb{Z}_p)$ and the normal representation at a fixed point.

   In the topological setting, the actions in the Rothenberg-Sondow theorem are conjugate iff the normal representations are the same. However, all known proofs of this claim are surprisingly difficult. Although Whitehead torsion is a topological invariant for closed manifolds, the situation is much more complicated for problems involving group actions and stratified spaces. Unfortunately, this topic cannot be discussed here, but see [Ste, Q4, We4].

   The group $\mathrm{Wh}(\mathbb{Z}_p)$ is free abelian of rank $(p-3)/2$; it is detected by taking the determinant of a representative matrix and mapping the group ring $\mathbb{Z}[\mathbb{Z}_p]$ to the ring of integers in the cyclotomic field associated to the p-th roots of unity. According to Dirichlet's unit theorem, the group of units of this number ring has rank $(p-3)/2$.

5. In general, there have been great strides in calculating $\mathrm{Wh}(\pi)$ for π finite (see [O]). We will see later that $\mathrm{Wh}(\pi)$ is conjecturally 0 for all torsion-free groups, and that there is even a conjectural picture of what $\mathrm{Wh}(\pi)$ "should" look like in general.

   This picture looks even stronger when combined with higher algebraic K-theory. Remarkably, the best general lower bounds we have for higher algebraic K-theory are based on the ideas developed for application in operator K-theory, namely, cyclic homology. See [BHM]. These results have implications for lower bounds on the size of the higher homotopy of diffeomorphism groups.

6. The h-cobordism theorem removes the possibility of any bundle theory, since bundles over an h-cobordism are determined by their restrictions to an end.[2] A key to understanding the role of bundles, unstably, is provided by Wall's $\pi - \pi$ theorem (as reformulated using work of Sullivan):

[1] This means that the boundary is embedded in the boundary.

[2] This claim is correct only when we discuss stable bundle theory; there is room for unstable information from the way in which the two boundary components destabilize the "same" stabler tangent bundle of the interior. This information does actually arise in the topological setting, and reflects a relationship between the destabilization of bundle theory and algebraic K-theory. Note that the "local structure" around points in the interior of the h-cobordism is the product of $\mathbb{R}$ and the local structure at a boundary point;

Theorem: Let M be a manifold with boundary such that $\partial M \to M$ induces an isomorphism of fundamental groups, and let $S(M) = \{(M', f) \mid f : (M', \partial M') \to (M, \partial M)$ is a simple homotopy equivalence of pairs$\}$/Cat isomorphism. There is then a classifying space depending on the category, denoted F/Cat, such that $S(M) = [M : \text{F/Cat}]$. This $S(M)$ is called the (Cat-) structure set of M.

If M is noncompact, one can analogously define $S^p(M)$, the proper structure set, using proper homotopy equivalences. This is all explained in [Wa, Br].

7. Much is known about the various F/Cat. For the duration we shall assume that Cat = Top. In that case, first of all, one has a complete analysis of F/Cat (due mainly to Sullivan, with an assist by Kirby and Siebenmann):

$$\begin{aligned}[M : \text{F/Top}] \otimes \mathbb{Z}_{(2)} &\cong H^4(M; \mathbb{Z}_{(2)}) \oplus H^8(M; \mathbb{Z}_{(2)}) \oplus H^{12}(M; \mathbb{Z}_{(2)}) \oplus \cdots \\ &\quad \oplus H^2(M; \mathbb{Z}_2) \oplus H^{10}(M; \mathbb{Z}_2) \oplus H^{14}(M; \mathbb{Z}_2) \oplus \cdots\end{aligned}$$

at 2, and away from 2,

$$[M : \text{F/Top}] \otimes \mathbb{Z}[1/2] \cong \text{KO}^0(M) \otimes \mathbb{Z}[1/2].$$

In the second formula, a structure is associated to (the “Poincaré dual” of) the difference of the signature operators on domain and range. In fact, all the formulas turn out to work much better in Poincaré dual form; the $\pi - \pi$ classification should then be given by $S(M^n) \cong H_n(M; \mathbb{L})$, where $\mathbb{L}$ is the spectrum whose homotopy type is determined by the above calculations in cohomology. The reason for this terminology will become clearer as we progress. The connection to the signature operator is hopefully suggestive as well. (For these topics, see [MM] and [RW2].)

8. The material in the previous sections gives rise to a complete analysis of $S(M)$ for M closed and simply-connected. Let $\widehat{M}$ denote M with a little open ball removed. Then $S(M) = S(\widehat{M})$ by the Poincaré conjecture. The latter satisfies the hypotheses of the $\pi - \pi$ theorem, and thus $S(M) \cong H_n(\widehat{M}; L)$. Concretely, up to finite indeterminacy, the structure set is determined by the differences between the Pontrjagin classes $p_i(M)$ for $4i < n$.

What about p_i when $4i = n$? The answer is that it is determined by the lower Pontrjagin classes. The reason is that the Hirzebruch signature theorem asserts that $\text{sign}(M^{4i}) = \langle L_i(M), [M] \rangle$. Here, the quantity $\text{sign}(M)$ is the signature of the inner product pairing on H^{2i} of the oriented manifold M, and L is a graded polynomial in the Pontrjagin classes of M. This formula has many remarkable consequences. For instance, Milnor used it to detect exotic spheres. However, for us, it first implies that a particular combination of Pontrjagin classes is homotopy invariant. As a second point, Hirzebruch's formula can be viewed as a simple application of the Atiyah-Singer index theorem [APSIII].

9. For general non-simply connected manifolds, there may exist further restrictions on the variation of the Pontrjagin classes, and there may exist more manifolds with the same tangential data. We shall deal with each of these possibilities one at a time. Although the complete story must necessarily involve interesting finite-order invariants, we shall concentrate on the $\otimes \mathbb{Q}$ story which, at our current level of ignorance, seems to be closely tied to analysis. Said slightly differently, the whole known and even conjectured story with $\otimes \mathbb{Q}$ can be explained analytically. However, no one has any direct approach to obtaining isomorphisms between L-theory and operator K-theory, and as we shall explain in the epilogue, this connection seems unlikely.

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equivalently, in the case of manifolds, the dimension of a manifold is one more than the dimension of its boundary. Hence the “tangential data” on the interior is ”stabler” than the data on the boundary.

10. The Novikov conjecture is the assertion that, if $f : M \to B\pi$ is a map, then the image of the Poincaré dual of the graded L-class of M in $\oplus H_{n-4k}(B\pi;\mathbb{Q})$ is an oriented homotopy invariant. Note that, for the homotopy equivalent manifold, one must use the obvious reference map to $B\pi$ obtained by composing the homotopy equivalence with f.

    For π trivial, this statement is a consequence of the Hirzebruch signature theorem. In fact, the Novikov conjecture is known for an extremely large class of groups at present. We will describe some of this work in the next lecture.

11. It is worth noting that the cases for which the Novikov conjecture is known are the only combinations of Pontrjagin classes that can be homotopy invariant. This claim can be proven axiomatically from the simply-connected case together with cobordism of manifolds and the $\pi-\pi$ theorem. However, we shall "take the high road," and use the surgery exact sequence, and work for simplicity in the topological category. In this venue, we assert that, for M a compact closed manifold of dimension at least 5, there is an exact sequence,

$$\cdots \to L_{n+1}(\pi_1) \to S(M) \to H_n(M,\mathbb{L}) \to L_n(\pi_1) \to \cdots$$

where the L are 4-periodic, purely algebraically defined groups, and covariantly functorial in $\pi_1 = \pi_1 M$.

If M has boundary and if one is working rel boundary then the same sequence holds. For manifolds with boundary, and for working not rel boundary, the sequence changes by the presence of relative homology groups and relative L-groups $L(\pi_1,\pi_1^\infty)$; the $\pi-\pi$ theorem then reduces to the statement that $L(\pi,\pi)=0$, which is perfectly obvious from the exact sequence of a pair (which indeed does hold in this setting).

We can do better by taking advantage of the periodicity.[3] Let M be an n-manifold, and define $S_k(M) = S(M \times D^j)$ for any j such that $n+j-k$ is divisible by 4. With that notation, the sequence becomes

$$\cdots \to L_{n+1}(\pi_1) \to S_n(M) \to H_n(M,\mathbb{L}) \to L_n(\pi_1) \to \cdots$$

(with obvious relative versions). With this notation, one can then say that the sequence is a covariantly functorial sequence of abelian groups and homomorphisms. The push-forward map on structures (elements of S-groups are called "structures") is closely related to the push-forward of elliptic operators of Atiyah and Singer [ASI] although defined very differently.

The functoriality implies that one can define $S_n(X)$ for any CW complex X just by taking the direct limit of $S_n(X^k)$ as X^k runs though any ascending union of sub-CW-complexes whose union is X. (Note that homology and L-theory both commute with direct limits.) Consequently, the map $H_n(M,\mathbb{L}) \to L_n(\pi_1)$ factors through the map $H_n(B\pi_1,\mathbb{L}) \to L_n(\pi_1)$. The latter is called the *assembly map*. For π trivial, the classification of simply-connected manifolds explained in Section 8 implies that the assembly map for a trivial group is an isomorphism. (Hence the homology theory introduced in section 7 has L-groups as its homotopy groups, explaining the source of the notation.) The groups $L_i(e) = \mathbb{Z}, 0, \mathbb{Z}_2, 0$ for $i = 0,1,2,3$ mod 4, respectively, exactly the homotopy groups of F/Top mentioned above.

[3] Periodicity is not quite true in the topological category: it can fail by a copy of $\mathbb{Z}$, if M is closed, and cannot fail if M has boundary. See [Ni], and see also [BFMW] for the geometric explanation and repair of this failure.

The commutativity of the diagram

$$\begin{tikzcd} H_n(M,\mathbb{L}) \arrow[r] \arrow[d] & L_n(\pi_1) \arrow[d] \\ H_n(B\pi_1,\mathbb{L}) \arrow[r] & L_n(\pi_1) \end{tikzcd}$$

quickly implies that the only possible restriction on the characteristic classes comes from the difference of the L-classes in $H_n(B\pi_1, \mathbb{L})$. Moreover, the homotopy invariance of the higher signatures is exactly equivalent to the rational injectivity of the assembly map.

12. A similar discussion applies to manifolds with boundary. We leave it to the reader, with references to [We1,3] for the impatient reader. For instance, the $\pi - \pi$ theorem implies that there are no homotopy invariant characteristic classes of $\pi - \pi$ manifolds. The extended higher signature conjecture would have posited the proper homotopy invariance of the L-class in $H_n(B\pi_1, B\pi_1^\infty) = 0$.

13. Although rational injectivity of the assembly map is conjectured to be universal, surjectivity is not. The simplest example of this notion comes from the Hirzebruch signature formula. Note that the right-hand side of the formula

$$\text{sign}(M) = \langle L(p^*(M)), [M]\rangle$$

is clearly multiplicative in coverings: if $N \to M$ is finite covering, the L-classes pull back, but the fundamental class is multiplied by the degree of the covering. This argument implies that, for closed manifolds, signature is multiplicative in coverings.

Note that, as a consequence, if M and M' are homotopy equivalent and cobordant by a cobordism V, then an obstruction to the homotopy equivalence being homotopic to a diffeomorphism can be obtained by gluing the boundary components of V together to obtain a Poincaré duality space, which might well not satisfy the multiplicativity of signature. In fact, this method can be extended further, as noted by [Wa] and [APS1, APS3], and underlies the proof of de Rham's theorem on lens spaces given in [AB]. Suppose for simplicity that M has fundamental group π, and so does the cobordism V mentioned above. Then the cohomology of the universal cover of V has a π action on it. The *equivariant signature* of this quadratic form can be shown to be a multiple of the regular representation; i.e. each character except for the one corresponding to the trivial element must vanish. Atiyah and Bott had computed these characters for the lens space situation in the course of their argument. The multiplicativity issue is equivalent to the vanishing of the average of these characters.

Remark: The multiplicativity invariant can be used even if the fundamental group is infinite: one must use von Neumann signature of the universal cover in place of ordinary signature. (The relevant multiplicativity is Atiyah's L^2 signature theorem.) This is the key point in the proof of the following flexibility theorem, perhaps one of the simplest general applications of analytic methods that does not yet have a purely topological proof:

Theorem [CW]: If M^{4k+3} has non-torsion-free fundamental group, $k > 0$, then $S(M)$ is infinite. (This theorem fails in all other dimensions, except 0 and 1 when the hypothesis is vacuous – at least if the Poincaré conjecture is true.)

14. The "trick" of the previous section can sometimes be turned into a method or an invariant, a so-called secondary invariant, even in situations where the manifolds are not (a priori known to be) cobordant. A first example arises in the situation of the previous section. If M is an odd-dimensional manifold with finite fundamental group, then some multiple sM, of it bounds a manifold with the same fundamental group. One can then consider the signature of the universal cover of that manifold, multiplied by $1/s$ to correct for the initial multiplication. (Signature can be defined for manifolds with boundary just by throwing away the singular part of the inner product.)

A much deeper way of accomplishing the same task, which applies in some circumstances where the cobordism group is not torsion, is due to [APS], who defined a real-valued invariant of odd-dimensional manifolds with finite-dimensional unitary representations of their fundamental groups. If the image of the representation is a finite group, it reduces to what we just considered above, but in general, it is much more subtle. Years ago, the first author conjectured that:

Conjecture: If $\pi_1 M$ is torsion-free, then for any unitary representation ρ, the Atiyah-Patodi-Singer invariant is homotopy invariant; in general it is an invariant up to a rational number.

The second statement was proven in [We2] as an application of known cases of the Novikov conjecture. In the original paper, it was shown to follow from the Borel conjecture. Keswani [Ke1] proved it for a class of groups, such as amenable groups, assuming a version of the Baum-Connes conjecture.

Cheeger and Gromov [ChG] considered the von Neumann analogue of this discussion. Mathai [Mat] made the analogous conjecture to the one above: that the Cheeger-Gromov invariant is homotopy invariant for manifolds with torsion-free fundamental group. Special cases are verified in [Mat, Ke2, CW, Ch2], in all cases using Novikov-like ideas. The flexibility result of the previous section is the converse to Mathai's conjecture.

Finally, we should mention the ideas of [Lo1] and [We1] which define "higher" versions of these secondary signature-type invariants in situations in which the Novikov conjectures provide for the existence of higher signatures to be definable (in a homotopy-invariant fashion). Unlike the classical secondary invariants, these ideas require some cohomological vanishing condition, rather like Reidemeister torsion. We will postpone further discussion of these subjects until section 17.

15. We have seen that the entire classification of simply-connected manifolds follows essentially from the Poincaré conjecture (Smale's theorem) and the formal structure of surgery theory. The same result is true for any fundamental group: understanding any manifold with that fundamental group well enough will determine the classification theory for all. The Borel conjecture, or topological rigidity conjecture, is the following:

Conjecture: If M is an aspherical manifold and $f : M' \to M$ is a homotopy equivalence, then f is homotopic to a homeomorphism.

In fact, it is reasonable to extend the conjecture to manifolds with boundary and homotopy equivalences f that are already homeomorphisms on the boundary. (Similarly, one can deal with proper homotopy equivalences between noncompact aspherical manifolds that are assumed to be homeomorphisms in the complement of some unspecified compact set.)

Notice that the Borel conjecture implies that $\mathrm{Wh}(\pi) = 0$ for the fundamental group of an aspherical manifold (exercise!). Note also that, by enlarging our perspective to include the noncompact case, the aggregate of π to which the conjecture applies is the set of countable groups of finite cohomological dimension. (In fact, it is pretty obvious, by direct matrix considerations, that one can remove the countability, if one so desires!) Furthermore, by feeding the problem into the surgery exact sequence, one obtains in addition the statement that $A : H_n(B\pi_1, \mathbb{L}) \to L_n(\pi_1)$ is an isomorphism for all n. Indeed, the Borel conjecture (for all n) is equivalent to the verity of these two assertions.

Much is known about the Borel conjecture; so far, no one knows of any counterexample to the claim that these algebraic assertions hold for all finite groups. (Note that for all groups with torsion, the map A fails to be a surjection by the flexibility theorem for n divisible by 4.)

16. It is probably worthwhile to discuss the motivation for the Borel conjecture and its variants. Reportedly, Borel asked this question in response to the theorems of Bieberbach and Mostow about the classification of flat and hyperbolic manifolds, respectively. In the first case, an isomorphism of fundamental groups gives an affine diffeomorphism between the manifolds and in the second (assuming the dimension is at least three) it gives an isometry (which is unique). Since symmetric manifolds of noncompact type are all aspherical, Borel suggested that perhaps this condition should be the topological abstraction of a symmetric space, and that, without assuming a metric condition, one should instead try for a homeomorphism.

In light of this suggestion, it is worthwhile to consider the noncompact version. For noncompact hyperbolic manifolds of finite volume (the nonuniform hyperbolic lattices), Mostow's rigidity theorem remains true (although its failure in dimension 2 is even more dramatic: the homotopy type does not even determine the proper homotopy type of the manifold) as was proven by Prasad. It might therefore seem reasonable (as was done in at least one ICM talk!) to suggest that Borel's conjecture could be extended to properly homotop a homotopy equivalence to a nonuniform lattice quotient to a homeomorphism. The following result that we proved jointly with Alex Lubotzky shows that this situation never occurs. (We shall give a different nonuniform rigidity theorem in Part II.)

Theorem: Suppose that Γ is a nonuniform irreducible arithmetic lattice in a semisimple Lie group G. Let K be the maximal compact subgroup of G. If $\text{rank}_{\mathbb{Q}}(\Gamma) > 2$, then there is a non-properly-rigid finite-sheeted cover of $\Gamma\backslash G/K$.

In part II we will explain why it is extremely likely that proper rigidity holds if $\text{rank}_{\mathbb{Q}}(\Gamma) = 1$ or 2. (If $\text{rank}_{\mathbb{Q}}(\Gamma) = 0$, then the lattice is cocompact by the well-known theorem of Borel and Harish-Chandra.)

The theorem is a simple combination of several deep ingredients. By the theory of Borel and Serre [BS], we find that any such $\Gamma\backslash G/K$ has a compactification as a $\pi - \pi$ manifold. According to Siebenmann's thesis, any manifold proper homotopy equivalent to it will have the same property. Using the h-cobordism theorem, any such manifold has a unique compactification so that the extension of the proper homotopy equivalence to the compactification is a simple homotopy equivalence. By the $\pi - \pi$ theorem, we have $S^p(\Gamma\backslash G/K) = [\Gamma\backslash G/K; \text{F/Top}]$. The calculations in section 7 then show that it suffices to prove that $H^2(\Gamma\backslash G/K; \mathbb{Z}_2)$ is nonzero, perhaps after replacing Γ by a subgroup of finite index, which we will still call Γ. Note that, under our assumptions, all irreducible lattices in Γ have vanishing first Betti number. If Γ were simple, this statement would follow from Kazhdan's property T. In general, it follows from superrigidity (see [Mar, Z]). By the universal coefficient theorem (recall that Γ is finitely generated), it thus suffices to find a lattice Γ such that $H_1(\Gamma; \mathbb{Z}_2)$ is nonzero. In fact, every infinite linear group has a subgroup with this property; this statement is equivalent to the theorem of [Lu] and [Weh] that every infinite linear group has an even-order quotient.

Remark: The theorem of Lubotzky is stronger and implies that one can force $S^p(\Gamma\backslash G/K)$ to have large rank by choosing the lattice more carefully. In fact, if the $\mathbb{R}$-rank is large enough and $\text{rank}_{\mathbb{Q}}(\Gamma) > 2$, then one can construct infinite structure sets with nontrivial elements detected by Pontrjagin classes (e.g. for $\text{SL}_n(\mathbb{Z})$ for n sufficiently large, using Borel's calculations). Unlike the elements constructed here, these elements do not die on passage to further finite-sheeted covers. Note that, for a product of three punctured surfaces, the proper rigidity conjecture is always false (for any cover), but is virtually true, in that any counterexample dies on passing to another finite cover!

17. We now shall consider a much more fruitful (but still false) conjecture suggested by the heuristic that lead to the Borel conjecture: the "equivariant Borel conjecture" or "equivariant topological rigidity conjecture." Notice that Mostow rigidity actually immediately implies the following seeming strengthening of itself.

Theorem: Suppose that M and N are hyperbolic manifolds, and $f : \pi_1 N \to \pi_1 M$ is an isomorphism which commutes with the representation of a group G on Out(π) induced by actions of G by isometries on M and N. Then there is a unique isometry between M and N (realizing f) which conjugates the G-actions to each other.

Mostow rigidity is the case of this theorem when G is trivial; on the other hand, since the isometry between M and N realizing any given group isomorphism is unique, it must automatically intertwine any actions by isometries that agree on fundamental groups. So let us now make another conjecture:

Conjecture: Suppose that G acts aspherically and tamely on a compact closed aspherical manifold M, and that $f : N \to M$ is an equivariant homotopy equivalence, then f is homotopic to a homeomorphism.

The condition that the action be tame means that one assumes that all components of all fixed point sets are, say, locally flatly embedded topological submanifolds, and the asphericality means that these components are all aspherical. This condition means that these spaces are the terminal objects in the category of spaces which are connected to a given one by equivariant 1-equivalences; i.e. one considers maps $X \to Y$ which induce isomorphisms $[K, X]^G \to [K, Y]^G$ for any $G-1$-complex, i.e. a 1-complex with a G-action. See [May].

In fact, this conjecture is false for several different reasons. However, it points us in the right direction. For one, its analytic analogue is the celebrated Baum-Connes conjecture. For a second, its "Novikov shadow" does seem to be true:

Conjecture (Equivariant Novikov conjecture, see [RW1]): Suppose that G acts tamely and aspherically on a finite-dimensional space X, and that $f : M \to X$ is an equivariant map. Then for any equivariant map $g : N \to M$ which is a homotopy equivalence, one has $f_* g_*(\Delta(N)) = f_*(\Delta(M))$ in $K_*^G(X)$, where Δ denotes the equivariant signature operator.

The hypothesis that G acts tamely is point-set theoretic; smoothness is certainly more than enough. For example, this conjecture holds whenever X is a symmetric space of noncompact type and G is a compact group of isometries of X. It is also worth noting that one can often build equivariant maps from any M with the appropriate fundamental group to this X using harmonic map techniques; see [RW1].

An analysis of this conjecture (for the case of discrete G) is that, if G is the "orbifold fundamental group of X," i.e. $1 \to \pi_1 X \to \Gamma \to G \to 1$ is exact, then $\mathrm{KO}_*^G(X) \otimes \mathbb{Z}[1/2]$ must inject into $L(\Gamma) \otimes \mathbb{Z}[1/2]$. In the next sections we will discuss more refined estimates of $L(\Gamma)$. Working rationally, we see where we went wrong in our understanding of $L(\Gamma)$. Before our estimate was $\mathrm{KO}_n(B\Gamma) \otimes \mathbb{Q} = \mathrm{KO}_n(X/G) \otimes \mathbb{Q}$, which differs a great deal from $\mathrm{KO}_n^G(X) \otimes \mathbb{Q}$ because of the fixed-point sets. If X is a point, we see that the representation theory of G enters, exactly as we saw before in our discussion of secondary invariants. In fact, most of the higher secondary invariants, when they are defined, take values in $\mathrm{KO}_{n+1}^G(EG, X) \otimes \mathbb{Q}$.

Remark: The conjecture that these refined lower bounds for L-theory hold universally would imply the infiniteness of structure sets proven above using L^2 signatures.

18. Unfortunately, the equivariant Borel conjecture is false. The first source of counterexamples discovered was related to the Nil's of algebraic K-theory [BHS, Wald1]. Soon thereafter analogous counterexamples were discovered based on Cappell's Unils [Ca1]. See [CK] for a discussion of these examples. The full explanation requires an understanding of equivariant h-cobordism and classification theorems; we cannot describe these topics here, but recommend the surveys [CaW, HW, We4].

Let us begin with the equivariant h-cobordism theorem. According to Steinberger and West's analysis (Quinn provided a more general version for all stratified spaces), one has an exact sequence:[4]

$$\cdots \to H_*(M/G; \mathrm{Wh}(G_m)) \to \mathrm{Wh}(G) \to \mathrm{Wh}^{top}(M/G \text{ rel sing}) \to H_*(M/G; K_0(G_m))$$

$$\to K_0(G) \to K_0^{top}(M/G \text{ rel sing}) \to H_*(M/G; K_{-1}(G_m)) \to \cdots$$

where G_m is the isotropy of the point m. Here the "rel sing" means that we are considering h-cobordisms which are already products on the singular set; note that, unlike the smooth case, this condition does not give us a neighborhood of the set on which it is a product. It is precisely that which is measured by the homology term.

Perhaps the connection between a Whitehead group and K_0 seems odd. This interaction is analogous to (and actually stems from) a phenomenon studied by Siebenmann, arising from his thesis. Siebenmann discovered that the h-cobordism theorem can be extended from the situation of compact manifolds to a wide range of noncompact manifolds. A condition that renders the statements much simpler is "fundamental group tameness," which asserts that there is an ascending exhausting sequence of compact sets $K_1, K_2, \dots$ in W, such that the maps $M \backslash K_1 \leftarrow M \backslash K_2 \leftarrow M \backslash K_3 \leftarrow \cdots$ are all 1-equivalences. Let us assume that W has one end (so these complements are all connected). Then we denote the common fundamental group of the complements by $\pi_1^\infty W$. According to Siebenmann, there is a map

$$\mathrm{Wh}^p(W) \to \mathrm{Wh}(\pi_1 W, \pi_1^\infty W)$$

which thus fits into an exact sequence

$$\cdots \to \mathrm{Wh}(\pi_1^\infty W) \to \mathrm{Wh}(\pi_1 W) \to \mathrm{Wh}^p(W) \to \mathrm{KO}(\pi_1^\infty W) \to \mathrm{KO}(\pi_1^\infty W) \to \cdots$$

Note that when the fixed set consists of isolated points, this exact sequence for Wh^p of the orbit space of the free part is the same as the Steinberger-West sequence.

If W is the interior of a compact manifold with boundary, then the target of the boundary map $\mathrm{Wh}^p(W) \to K_0(\pi_1^\infty W)$ measures the obstruction to completing the h-cobordism as a manifold with corners. The homology term is analogous to a controlled K_0 or Wh; we will return to controlled algebra in Part II. In the Whitehead story, it turns out that Wh^{top} decomposes into a sum of terms, one for each stratum of M/G, each of the form $\mathrm{Wh}^{top}(Z/H \text{rel sing})$ for some Z and some H. For surgery theory, this decomposition does not hold, and the strata interact in a much more interesting way.

In any case, let us now consider some particularly simple equivariantly aspherical manifolds, and understand what is implied by the vanishing of Wh^{top}. Let $M = D^n$ with a linear action. Then the homology would be concentrated at the origin. The map $H_*(M/G; \mathrm{Wh}(G_m)) \to \mathrm{Wh}(G)$ is an

[4] For the development of of such homology groups, see [Q2] and the appendix of [We4] for homology with coefficients in a cosheaf of spectra. Note that it is an analogue of generalized cohomology theories and of sheaf cohomology.

isomorphism (and similarly for K_0), and indeed $\mathrm{Wh}^{top}(M) = 0$. Now let us consider $M = S^1 \times D^n$. Again the homology is concentrated entirely on the singular part, and we have

$$H_*(S^1; \mathrm{Wh}(G)) \cong \mathrm{Wh}(G) \times K_0(G) \to \mathrm{Wh}(\mathbb{Z} \times G).$$

This map is indeed a split injection, but it is not an isomorphism. The cokernel is $\mathrm{Nil}(G) \times \mathrm{Nil}(G)$ according to the "fundamental theorem of algebraic K-theory" [Ba]. These Nil groups are rather mysterious. A general theorem of Farrell show that Nil is infinitely generated if it is nontrivial. Some calculations can be found in [BM] and [CdS].

Similarly, in L-theory, the equivariant Borel conjecture would imply calculational results about the L-theory of, say, linear groups. The map $H_*(M/G; L(G_m)) \to L(G)$ should be an isomorphism when M is aspherical. (Incidentally, away from 2, the left-hand side can be identified with $\mathrm{KO}^G[1/2]$.) It is also not so hard to change the context of all of this discussion from finite quotients of aspherical manifolds to proper actions on contractible manifolds. However, these conjectures were already disproved by Cappell's results on the infinite dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2$. Cappell showed that $L_2(\mathbb{Z}_2 * \mathbb{Z}_2)$ is not just a sum of copies of $L_2(\mathbb{Z}_2)$ concentrated at fixed points, but rather that there is another infinitely generated summand that is not in the image of the relevant homology group.

These conjectures can be somewhat rehabilitated by considering the properties of Nil and Unil. For instance, one can ask about other rings besides integral group rings. The conjectures then lose some of their geometric immediacy, but with $\mathbb{Q}$, for instance, they stand a chance of being true. (For instance, whenever $1/2$ is in the coefficient ring, Cappell's Unils vanish identically, and there is no need for any further corrections to the isomorphism conjecture.) Just as the "fundamental theorem of algebraic K-theory" is true for all rings, it is very worthwhile to understand to what extent the purported calculations apply to general rings, even to the point of mere split injectivity. One might mention at this point the wonderful result of Bokstedt, Hsiang and Madsen affirming the algebraic K-theoretic version of injectivity of the usual assembly map for $\mathbb{Z}$, after tensoring with the rationals, for groups with finitely generated integral homology [BHM]. Unfortunately, their method does not apply to other rings, and does not directly imply anything about the algebraic K-theoretic injectivity statement raised here.

19. There is another more geometric reason why the equivariant Borel conjecture fails; the reasons are orthogonal to the algebraic problems discussed in the previous section, but are, undoubtedly,[5] related to pseudoisotopy theory. This second failure occurs when the gap hypothesis does not hold.[6] Again, the construction of the counterexamples and their classification would take us rather far afield, but it seems worth mentioning one simple example: a crystallographic group.

Suppose that one looks at an action of $\mathbb{Z}_p$ on $T^{p-3} \times T^p$, where the action on the first coordinates is trivial, and on the second set is by permutation. The fixed point set is a T^{p-2} in T^{2p-3} exactly at the edge of dimensions for which it is possible for homotopic embeddings not to be isotopic. In fact, it is quite easy to build nonisotopic embeddings: take a curve in $\pi_1(T^{2p-3})$ and push a small sheet of the T^{p-2} around that curve and then link this little sheet to the original T^{p-2} some number of times. This construction does not completely determine the curve. Using the opposite linking, one can replace

[5] This word brazenly advertises that we are aware of no direct connections.

[6] In fact, there are general theorems of Shirokova [Shi] which assert that the equivariant Borel conjecture is always false (under very weak assumptions) whenever the gap hypothesis fails. This gap hypothesis is the bane of the classification theory of group actions; it assumes that, whenever M^H lies in M^K, either these fixed sets coincide or one is somewhat less than half the dimension of the other. Necessary in the establishment of the foundations of equivariant surgery theory (see Memoirs of the AMS by Dovermann and Petire and by Dovermann and Rothenberg), the condition is needed to allow for surgeries performed inductively over the strata. It is important to realize that the gap hypothesis is usually assumed to make progress, not at all because it is natural or generally true.

a curve by its opposite; also curves that can be homotoped into T^{p-2} do not change the isotopy class of the embedding. But in essence one produces isotopy classes of embeddings of T^{p-2} isotopic to the original embedding (see [Shi]), one for each element in $\mathbb{Z}[\mathbb{Z}^{(p-1)*}]^{\mathbb{Z}_2}$.

Cappell and the second author showed [CaW] in this special case that an embedding is the fixed set of some $\mathbb{Z}_p$-action equivariantly homotopy equivalent to the affine action iff the new embedding is isotopic to its translate under the $\mathbb{Z}_p$ action. Furthermore, the action is unique up to conjugacy. Note that the counterexamples exist even rationally; they do display a nilpotency. Any particular action is conjugate to the affine action after passing to a cover. Passing to any large enough cover, we find that the curves used for modifying the embedding no longer go around, and thus do not change the embedding. More generally, there are connections to embedding theory, but not quite as precise as they are in this special case [We1].

20. Farrell and Jones have suggested another very general way to handle the failure of these algebraic assembly maps to be an isomorphism. Essentially the idea is the following: we could have been led to our previous isomorphism conjecture by a somewhat different line of reasoning.

Observing that the assembly map $A : H_*(B\pi; \mathbb{L}) \to L_*(\pi)$ is not an isomorphism for groups with torsion, we could have looked for "the universal version of an assembly map that does not oversimplify $L(\pi)$ for π finite." For each Γ we might consider $\underline{E}\Gamma$, which has an equivariant map from $E\Gamma \to \underline{E}\Gamma$, and build an assembly map $H_*(\underline{E}\Gamma/\Gamma; L(\Gamma_x)) \to L(\Gamma)$. (The various Γ_x will run over finite subgroups of Γ.) Now that we see that even this modified conjecture fails for groups like $\mathbb{Z} \times \pi$, for π finite, and $\mathbb{Z}_2 * \mathbb{Z}_2$, they suggest reiterating the process, but now with respect to the "virtually cyclic groups," i.e. the groups with cyclic subgroups of finite index. Thus one builds a more complicated classifying space E^Γ, on which Γ acts, and uses it for an assembly map. One should be somewhat careful, because the action cannot be proper to give us the requisite infinite groups as isotropy, but it is quite simple to build the correct thing simplicially. We refer the interested reader to [FJ2, DL] for more information.

21. It is worth making one more remark before closing this general lecture (i.e. one devoid of information about the conjectures themselves) about the connection to index theory and the geometric implications thereof.

This book is devoted to analogues in index theory of the Novikov conjecture and the Borel conjecture, indeed in all of the versions discussed above and in the ones to be discussed in Lecture 2. There is no need for the Farrell-Jones isomorphism ideas, because for virtually cyclic groups the Baum-Connes conjecture is true, unlike its Borel cousin. The topological side of these issues, for the moment, has additional complications arising from deep arithmetic connections. (The L-theory of finite groups, for instance, has a beautiful and important arithmetic side not visible in the theory of C^*-algebras). On the other hand, the analysis has beautiful connections to representation theory (some discussion on this subject will spill off into the Epilogue) and other geometric applications through other operators besides the signature operator.

These other applications can also suggest a variety of problems and methods. Two of these merit at least a mention here, although we do not have the space to develop them fully. The first is the work that a number of people have done on the (generalized) Hopf conjecture: that for any aspherical $2n$-manifold, the Euler characteristic is 0 or of sign $(-1)^n$. Methods of L^2 index theory applied to the de Rham (and Dolbeault) complex have given positive results in some spaces with negative curvature and Kähler structure (see [Gr]). The second problem, now known to be very closely tied to the Baum-Connes conjecture, is the characterization of the closed[7] manifolds which have metrics of positive scalar curvature.

The connection between index theory and the positive scalar curvature problem already appears in the Annals papers of Atiyah and Singer [AS]; they prove in the same paper the Hirzebruch signature formula using the index theorem and a vanishing theorem for the $\widehat{A}$-genus of a spin manifold with positive scalar curvature (based on a key calculation of Lichnerowicz). Combining ideas that are intimately related to (partial results on) the Novikov conjecture, one can obtain information on the non-simply connected case. We recommend the papers [GL1,2, Ros1,2,3, Sto] and the forthcoming monograph by Rosenberg and Stolz that shows that a "stable" version of the positive scalar curvature problem can be completely solved if the Baum-Connes conjecture were true, or even just the (easier) injectivity half.

Lecture Two: K-theory and Topology

This lecture is an introduction to some of the topological methods that have been applied to the conjectures made in the previous lecture. Unlike that lecture which explained how K-theory contributes to topology, this one studies contributions that topology makes to K-theory and L-theory.

1. The same general technique used to prove the h-cobordism theorem (handlebody theory) was subsequently applied by a number of researchers to a host of other problems, which in light of surgery theory imply solutions to Novikov and Borel conjectures in special cases. Here are some of those problems:
   (a) Putting a boundary on a noncompact manifold: Suppose that W is a noncompact manifold. When is W the interior of a compact manifold with boundary? Aside from homotopical or homological conditions at infinity, the answer is regulated by $K_0(\pi_1^\infty W)$. This idea was Siebenmann's thesis [Si1]. A nice special case due earlier to [BLL] is that, if W is simply-connected at infinity, then W can be compactified iff its integral homology is finitely generated. You might want to go back now and review the proper h-cobordism discussion from the last lecture.
   (b) Fibering over a circle: Without loss of generality, suppose that one has a surjection from $\pi_1 M \to \mathbb{Z}$. When is this map induced from a fiber bundle structure on M over the circle? The obvious necessary condition is that the induced infinite cyclic cover of M should have some finiteness properties. Ultimately, the result is determined by $\mathrm{Wh}(\pi_1 M)$. The history here is somewhat complicated; see Farrell's thesis for an analysis of the problem as a sequence of obstructions in terms of pieces of $\mathrm{Wh}(\pi_1 M)$; Siebenmann [Si3] gave a complete "one step" analysis of the problem. [F] sketches a very simple proof based on Siebenmann's thesis.

      Remark: If one lightens the demand on the fibration to be an approximate fibration (see, e.g. [HTW]), then the obstruction to an approximate fibration over S^1 lies entirely in the Nil piece of $\mathrm{Wh}(\pi_1 M)$. In this form, a slightly strong form of the Borel conjecture can be stated as follows:

      Conjecture: Suppose that M is a manifold and V is the cover of M induced by a homomorphism $\pi_1 M \to \Gamma$, where Γ is a group with $B\Gamma$ a finite complex. Then there is an aspherical homology manifold Z and an approximate fibration $M \to Z$ iff V is homotopically finite, and an obstruction involving the various $\mathrm{Nil}(\pi_1 V)$ vanishes.

[7] This connection extends rather further into the setting of noncompact manifolds, as we will discuss in Part II. Other noncompact instances will be mentioned later.

Note that this conjecture implies that $B\Gamma$ is automatically a Poincaré complex; this implication can be verified directly. Otherwise, the space V is never finitely dominated.[8] When V is simply-connected, this conjecture boils down to the Borel conjecture for the group Γ (see the discussion in Chapter 13 of [We4] and also the introduction to [HTWW]). Also, as a result of [WW], if one wants to avoid discussion of approximate fibrations, in the special case that Z is a manifold, one can decide to allow $T \times M$ to fiber over Z (for some torus factor), and then one can remove the Nil obstruction as well.

(c) Splitting theorems: Here one has a homotopy equivalence $f : M' \to M$ and a codimension one submanifold N of M; the problem is to homotop f to a map, still called f, such that f is transverse to N, and $f^{-1}(N)$ is homotopy equivalent to N (mapped to one another by f). The ultimate theorem in this direction is Cappell's splitting theorem, which applies whenever $\pi_1 N$ injects into $\pi_1 M$ and the normal bundle of N is trivial.

Earlier partial results are due to Wall, Farrell, Farrell-Hsiang, Lee and others.

Cappell [Ca1] gave very useful conditions under which $\tau(f)$, the Whitehead torsion of the map, determines the obstruction. However, in general this claim does not hold: in [Ca2] he gave infinitely many PL manifolds homotopy equivalent to $\mathbb{RP}^{4k+1}\#\mathbb{RP}^{4k+1}$ that are not connected sums. This example is responsible for some of the instances of non-rigid affine crystallographic group actions on Euclidean space discussed in the last lecture.

Note that the fibering theorem gives some situations in which one can analyze the splitting problem, and one can show in fact (using some surgery theory) that the splitting theorem and the fibering problem are equivalent for the class of groups that arise in the latter problem: The fibering problem reduces to an analysis groups that act simplicially on the line, and the splitting theorem to those that act on some tree.

2. The translation of fibering and splitting theorems was done first by Shaneson in his thesis [Sha]; see also Wall's book [Wa]. This translation led to the first proofs of the Borel conjecture for tori by Hsiang-Shaneson and Wall (the same proof works verbatim for poly-$\mathbb{Z}$ groups); Farrell and Hsiang had earlier given a proof of the Novikov conjecture for the free abelian case. Cappell's paper [Ca1] gives the Mayer-Vietoris sequence in L-theory for groups acting on trees associated to his splitting theorem. The corresponding theorems in algebraic K-theory are due to Waldhausen [Wald1] and in operator K-theory to Pimsner [Pi]. However, as is now extremely well known, most interesting groups do not act at all on trees. (See Serre [Se] for an early example.)

3. The vanishing of algebraic K-groups and the Borel conjectures were next proved for the class of flat and almost flat manifolds in a very beautiful and influential paper of [FH]. This paper combined a variant of Brauer's induction theory from classical representation theory, due to Dress [Dr], with controlled topology methods. These methods could have been adapted (more easily, in fact) to index theory, but there never seemed to be a need for it. They were however applied successfully to crystallographic groups with torsion in algebraic K-theory by [Q3] and to L-theory by Yamasaki [Ya]. These papers were very influential in formulating the cruder isomorphism conjectures mentioned in section 17 of Lecture I. (A more perspicacious mathematical community could have done so on the basis of thinking carefully about proper actions on trees, which can be analyzed on the basis of the theorems of Waldhausen and Cappell.)

[8] According to Wall, this condition is equivalent to a chain complex condition.

4. To develop controlled topology one must redo all of the classical topology problems such as those mentioned in section 1 (for example, putting boundaries on open manifolds), but in addition keep track of the size of these constructions in some auxiliary space. Here is the classical example.

   Theorem [CF]: Suppose that M is a compact manifold. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that, if $f : N \to M$ is a δ-controlled homotopy equivalence, then it is ϵ-homotopic to a homeomorphism.

   Now for the definitions. A δ-*controlled homotopy equivalence* is a map $f : N \to M$, equipped with a map $g : M \to N$, so that the composites fg and gf are homotopic to the identity by homotopies H and H', such that the tracks (i.e. the images of $(H')^{-1}(p,t)$ as t varies, for any specific p) of all of these homotopies (perhaps pushed using f) in M have diameter less than δ. A similar definition holds for ϵ-homotopy.

   The result stated (called the α-approximation theorem) is an example of a rigidity theorem. While it clarifies the idea of control, the following theorem of Quinn [Q1,2] separates the geometric problem from the control and also has obstructions, and thus give a better feel for the subject.

   Theorem: (Controlled h-cobordism theorem). Let X be a finite-dimensional ANR (e.g. a polyhedron). Then for all $\epsilon > 0$ there is a $\delta > 0$ such that, if $f : M^n \to X$ is a map with all "local fundamental groups $= \pi$" (e.g. if there is a map $M \to B\pi$ which when restricted to any fiber of f is an isomorphism on fundamental group), $n > 4$, then any δ-h-cobordism with boundary M defines an element in $H_0(X; \mathrm{Wh}(\pi))$; this element vanishes iff the h-cobordism is ϵ-homeomorphic over X to $Mx[0,1]$. Moreover, every element of this group arises from some h-cobordism.

   Notice two extremes: If X is a point, this result is the classical h-cobordism discussed in I.3. If $M = X$ then this result gives a metric criterion (due to Chapman and Ferry) that can be used to produce product structures, since $\mathrm{Wh}(e) = K_0(e) = K_{-i}(e) = 0$. This result includes the celebrated result of Chapman that the Whitehead torsion of a homeomorphism vanishes. Note that we need all the negative K-groups because Wh is a spectrum, so all of its homotopy contribute to the homology groups. The main theorems of controlled topology assert that various types of controlled groups are actually groups that are parts of homology theories. See [CF,Q1,2,4,FP,We4] for more information.

5. We give a cheating application of the α-approximation theorem to rigidity phenomena. Suppose that $f : M \to T$ is a homotopy equivalence to a torus. Now, pass to a large finite cover; the target is still a torus, which we identify with the original one by the obvious affine diffeomorphism. Now we have a new map which is an α-approximation. Thus, all sufficiently large covers of M are tori. Unfortunately, this proof is somewhat circular (at least for the original proof of α-approximation which used the classification of homotopy tori.) However, a slight modification of this argument shows that any embedding of the torus in another torus of codimension exceeding two, homotopic to an affine embedding, is isotopic to the affine embedding in all sufficiently large covers. It also can be used to show that any sufficiently large cover of a homotopy affine G-torus is affine (see [Ste]). To reiterate, as we saw before, the counterexamples to equivariant Borel mentioned in I.18 and I.19 die on passage to covers. Controlled topology implies that they all do.

6. There are, by now, a number of other versions of control in the literature, which, while fun for the experts, can be somewhat bewildering to the beginner. Some of these are: bounded control [Ped1, AM, MR, FP, HTW], continuous control at infinity [ACFP, Ped2], and foliated control [FJ3,4]. These theorems have all enjoyed applications to rigidity and to the Novikov conjecture. They are also important

in other topological problems. To give the flavor of one of these variants, let us discuss the bounded theory.

Definition: Let X be a metric space. A *space over* X is a space M equipped with a map $f : M \to X$. The map f need not be continuous, but it is usually important that it be proper. A map between spaces (M, f) and (N, g) over X is a continuous map $h : M \to N$ such that $d_X(f(m), gh(m)) < C$ for some C. Since C can vary, this construction forms a category.

If X has bounded diameter, this category is essentially equivalent to the usual category of spaces and maps. But, things heat up a lot when X is as simple as the real line or Euclidean space. Note that it is easy to define the homotopy category over X, and thus notions of h-cobordism over X, homotopy equivalence and homeomorphism can all be defined "over X." Note also that the all-important fundamental group must be generalized in this setting. Without any additional hypothesis, this generalization can be complicated (see [A]), but the following condition is often sufficient, especially for problems involving torsion-free groups:

Definition: The *fundamental group of* (M, f) *over* X is π, if there are constants C and D, and a map $u : M \to B\pi$ such that, for all $x \in X$, the image of π_1 of the inverse image of the ball of radius C about x inside the inverse image of the ball of radius D is isomorphic to π via the map u.

For simplicity we will assume that this condition holds, unless otherwise stated.

Example: If M is a space with fundamental group Γ, then its universal cover is a space over Γ, where Γ is given its word metric. It is in fact simply-connected over Γ. (A typical "bad example" would be an irregular cover, e.g. the cover of M corresponding to finite subgroups (which plays an important role in understanding groups with torsion.)

Remark: The analogue in this setting of being contractible is being uniformly contractible; i.e there is a function f such that, for all x in X and all C, the ball $B_x(C)$ is nullhomotopic in $B_x(f(C))$. Similarly, the notion of uniform asphericality requires that the map from the cover of $B_x(C)$ into the universal cover of $B_x(f(C))$ be nullhomotopic.[9]

These spaces are the terminal objects in the subcategory of spaces with the same "bounded 1-type" of a given one. By analogy, we shall be interested in their rigidity properties.

7. It is worth pondering the theory in some detail when $X = \mathbb{R}^n$. First, let us consider the Novikov conjecture in this setting:

   Theorem: Let M be a manifold with a proper map π to $\mathbb{R}^n$ and $f : N \to M$ be a homotopy equivalence over $\mathbb{R}^n$. Give N the structure of a space over $\mathbb{R}^n$ by using πf. Then $\text{sig}(\pi^{-1}(0)) = \text{sig}(f^{-1}\pi^{-1}(0))$.

   Here, we are assuming that 0 is a regular value of π and πf. This theorem readily implies Novikov's theorem about the topological invariance of rational Pontrjagin classes [No]. See [FW2].

8. The following geometric result of Chapman is sufficient for proving a number of bounded Borel conjecture results. After one develops bounded Whitehead theory and surgery, it implies calculations of K-groups and L-groups, ones which can be done independently algebraically (see [PW, FP]). Even after they are proven, the following theorem still feels "greater than the sum of its parts."

   Theorem [Ch]: Suppose that $N \to M = V \times \mathbb{R}^n$ is a bounded homotopy equivalence, where V is some compact manifold. Then there is some manifold Z homotopy equivalent to $V \times T^n$ whose infinite

[9] We assume that the maps induced by inclusions of balls in one another are injections on fundamental group.

abelian cover is N; moreover, the manifold Z is unique up to homeomorphism if we insist that it be "transfer invariant," i.e. homeomorphic to its own finite-sheeted covers that are induced from the torus.

The K-theoretic version would be that $\mathrm{Wh}^{bdd}(V \times \mathbb{R}^n) = K_{1-n}(\pi_1(V))$ and that $L_k^{bdd}(V \times \mathbb{R}^n) = L_{k-n}^{-n}(V)$. In the second case, the dimension of the L-group is shifted (remember they are 4-periodic, so negative L-theory is nothing to fear), and the superscript "decoration" is also shifted. See [Sha, PR, Ra, We4, WW] for some discussion of this topic.

9. Now that we have a bounded version of the Borel conjecture, we can repair the aesthetic defect uncovered in I.18: we can give a topological rigidity analog of Mostow rigidity for nonuniform lattices. For noncompact arithmetic manifolds $M = \Gamma\backslash G/K$, where G is a real connected linear Lie group and K its maximal compact subgroup, the slight strengthening of Siegel's conjecture proven in [J] provides the following picture from reduction theory. For each such M there is a compact polyhedron P and a Lipschitz map $\pi : M \to cP$ from M to the open cone on P such that (1) every point inverse deform retracts to an arithmetic manifold, (2) the map π respects the radial direction, and (3) all point inverses have uniformly bounded size. See Chang [C] for further discussion.

   In [FJ1] Farrell and Jones show topological rigidity of these arithmetic homogeneous spaces relative to the ends. On the contrary, if $\mathrm{rank}_{\mathbb{Q}}(\Gamma) > 2$, then M may not be properly rigid, as discussed in the remark of I.17. The following theorem asserts that M is topolgically rigid in the category of continuous coarsely Lipschitz maps.

   Theorem: Let $M = \Gamma\backslash G/K$ be a manifold for which Γ is an arithmetic lattice in a real connected linear Lie group G. Endow M with the associated metric. If $f : M' \to M$ is a bounded homotopy equivalence, then f is boundedly homotopic to a homeomorphism.

   To see that the reduction theory implies the vanishing of $S^{bdd}(M)$, one appeals to the bounded surgery exact sequence:

$$H_{n+1}(M; \mathbf{L}(e)) \to L_{n+1}^{\mathrm{bdd}}(M) \to S^{\mathrm{bdd}}(M) \to H_n(M; \mathbf{L}(e)) \to L_n^{\mathrm{bdd}}(M).$$

   We note that the radial direction of cP can be scaled to increase control arbitrarily, and that all the fundamental groups arising in the point inverses $\pi^{-1}(*)$ are K-flat by [FJ1]. These two ingredients give us an isomorphism $L_*^{\mathrm{bdd}}(M) \cong H_*(cP; \mathbf{L}(\pi^{-1}(*)))$. Given the Leray spectral sequence for π and the stalkwise equivalence of L-cosheaves, one also has the identification $H_*(M; \mathbf{L}(e)) \cong H_*(cP; \mathbf{L}(\pi^{-1}(*)))$. These isomorphisms give the required vanishing of $S^{\mathrm{bdd}}(M)$.

   Remark: A C^*-algebraic analogue of this calculation is relevant to the question of whether M has a metric of positive scalar curvature in its natural coarse quasi-isometry class (see following section). The unresolved state of the Baum-Connes conjecture for lattices prevents from from repeating the above argument in that setting.

10. It is now well recognized that the original approach by [GL2] and [SY] proving that no compact manifold of nonpositive sectional curvature can be endowed with a metric of positive scalar curvature is actually based on a restriction on the coarse quasi-isometry type of complete noncompact manifolds. Block and Weinberger [BW] investigate the problem of complete metrics for noncompact symmetric spaces when no quasiisometry conditions are imposed. In particular they show the following:

Theorem: Let G be a semisimple Lie group and consider the double quotient $M \equiv \Gamma\backslash G/K$ for Γ irreducible in G. Then M can be endowed with a complete metric of positive scalar curvature if and only if Γ is an arithmetic group of with $\text{rank}_{\mathbb{Q}}\Gamma \geq 3$.

In fact, in the case of $\text{rank}_{\mathbb{Q}}\Gamma \leq 2$, one cannot impose upon M a metric of uniformly positive scalar curvature even in the complement of a compact set. However, in the cases for which such complete metrics are constructed, they always exhibit the coarse quasiisometry type of a ray.

In the context of the relative assembly map $A : H_*(B\pi, B\pi^\infty, \mathbb{L}) \to L(\pi, \pi^\infty)$ and the classifying map $f : (M, M_\infty) \to (B\pi, B\pi^\infty)$, one might expect that the obstruction for complete positive scalar curvature on a spin manifold M is given by the image $f_*[D_M]$ of the Dirac operator in $KO_n(B\pi, B\pi^\infty)$ instead of the signature class in $H_*(B\pi, B\pi^\infty, \mathbb{L})$. One could reasonably conjecture that a complete spin manifold with uniformly positive scalar curvature satisfies $f_*[D_M] = 0$ in $KO_n(B\pi, B\pi^\infty)$ if $\pi_1(M)$ and $\pi_1^\infty(M)$ are both torsion-free.

However, the standard methods in L-theory fail in the K-theoretic framework because there is no assembly map from $KO_n(B\pi, B\pi^\infty)$ to the relative K-theory of some appropriate pair of C^*-algebras which might reasonably be an isomorphism for torsion-free groups. For the case of $\text{rank}_{\mathbb{Q}}\Gamma = 2$ considered in [BW], an alternate route was found (the cases of $\text{rank}_{\mathbb{Q}}\Gamma \leq 1$ are covered by [BH] and [GL2]): the verification that $f_*[D_M]$ vanishes in $KO_{n-1}(B\pi^\infty)$ under the assumption that suitable Novikov-type conjectures hold for the group π^∞.

The results of [BW] do not settle whether these uniformly positive curvature metrics on $\Gamma\backslash G/K$ can be chosen to be (a) quasiisometric (i.e. uniformly bi-Lipschitz) to the original metric inherited from G or (b) of bounded geometry in the sense of having bounded curvature and volume. The first author proved the former negatively in [C] by identifying a coarse obstruction of Dirac type in the group $K_*(C_*(M, \pi))$, where $C_*(M, \pi)$ is a generalized Roe algebra of locally compact operators on $\widetilde{M}$ whose propagation is controlled by the projection map $\pi : \widetilde{M} \to M$. This algebra encodes not only the coarse behavior of M but also its local geometry.

11. The principle of descent was first formulated explicitly in [FW2], although it appears, somewhat implicitly in [GL2, Ka, FW1, C, CP] as well. The paper of Gromov and Lawson is especially nice from this point of view, in that they explicitly suggest the use of a families form of a non-compact index theorem to deduce Gromov-Lawson conjecture type results (for manifolds of positive scalar curvature). The principle of descent is, in genrral, a vehicle for translating bounded Borel or Baum-Connes conjecture type results from the universal cover of a manifold, to deduce Novikov conjecture type results for the manifold itself. This principle remains a powerful tool and is exploited in the most recent exciting advances in the subject (see, e.g. [Yu1, Tu2, HR]).

The bounded Novikov conjecture states that, if X is uniformly contractible and M is a manifold over X, then $f_*(L(M) \cap [M]) \in H_*^{\ell f}(X; \mathbb{Q})$ is a bounded homotopy invariant. Equivalently, if Γ is the fundamental group of M, then the bounded assembly map

$$A^{bdd} : H_*^{lf}(E\Gamma; \mathbb{L}(e)) \to \mathbb{L}^{bdd}(E\Gamma)$$

is a split injection. Here $\mathbb{L}^{bdd}(E\Gamma)$ is the spectrum whose 0-th space is given by a simplicial model for which the n-simplices are n-ad surgery problems on k-manifolds together with a proper coarse map to Γ with the word metric. The map is on the level of the space of sections of assembly maps associated to the fibration $E \times_\Gamma E \to B\Gamma$ to a twisted generalized cohomology.

Assuming the split injectivity of A^{bdd}, we consider the following commutative diagram:

$$\begin{tikzcd} H_*(B\Gamma;\mathbb{L}(e)) \arrow[r, "A"] \arrow[d, "\cong"] & \mathbb{L}_*(\mathbb{Z}\Gamma) \arrow[d, "FBT"] \\ H^0(B\Gamma; H_*^{\ell f}(\mathbb{R}^n;\mathbb{L}(e))) \arrow[r, "A^{bdd}"] & H^0(B\Gamma;\mathbb{L}^{bdd}(\mathbb{R}^n)) \end{tikzcd}$$

The right-hand "family bounded transfer map" is a composite $\mathbb{L}_*(\mathbb{Z}\Gamma) \to H^0(B\Gamma;\mathbb{L}^{bdd}(E\Gamma)) \to H^0(B\Gamma;\mathbb{L}^{bdd}(\mathbb{R}^n))$. The left-hand vertical isomorphism arises when $B\Gamma$ is a finite complex from Spanier-Whitehead duality and the proper homotopy equivalence of the map $E\Gamma \to \mathbb{R}^n$. It too is a family bounded transfer map, for which, at each point x of $B\Gamma$, one lifts a cycle to the universal cover $E\Gamma$ based at x. The splitting of A^{bdd} clearly induces a splitting of A, and the descent argument is complete.

The principle of descent is also instrumental in deducing the (analytic) Novikov Conjecture from the coarse Baum-Connes Conjecture. The latter states that, for any bounded geometry space X, the coarse assembly map $A_\infty : KX_*(X) \to K_*(C^*X)$ is an isomorphism. Here KX_* denotes the coarse homology theory corresponding to K-homology. See [Roe] for more details.

Epilogue

In this epilogue, we would like to mention some issues that there was no time to discuss during those lectures. For the most part, the ideas discussed above provide quite close parallels (at least at the level of conjecture) between topology and index theory. There are several areas where the subjects have diverged that create new opportunity for further developments in one subject or the other.

1. KK (and E). In index theory there is a powerful computational calculus which builds two-way maps between relevant groups. So far, no analogous flexible theory has been developed in topology. Besides the sad conclusion that beautiful results like those of [HK, Tu1, Yu1] are not yet known on the topological side (let alone in algebraic K-theory, other coefficient rings, twistings, etc.), even the simple curvature calculations of [Yu1] which give a rational counterexample to a coarse version of the Novikov conjecture for a metric space without bounded geometry cannot be copied in topology. Thus, we have very little information about how the epsilons in controlled topology depend on dimension.

2. In topology, however, there are a number of subtle arithmetic issues that don't arise in index theory. Some of these are associated to "decorations"[10] (see [Sha]), Nil and Unil (see [Ba, Ca]). The latter shows that very routine version of Baum-Connes type conjectures in topology are false even for the infinite dihedral group (which is crystallographic and hence amenable).

   However, injectivity statements still have a reasonable chance of being correct. For instance, it seems that the object $\lim(L^{bdd}(V))$, where V runs through the finite-dimensional subspaces of a Hilbert space (V included in W gives rise to a map between the bounded L-theories by taking the product, used in defining the directed system) should arise in geometrizing [HK]. A dreamer could hope that one can do this geometrization topologically using a more complicated category (more arrows connecting subspaces to one another) for other Banach spaces. But, at the moment, one cannot even recover the analytically proven results for groups which embed uniformly in Hilbert space.

[10] Decorations are superscripts adorning L-groups, and modify their definitions by restricting or refining their precise defintion using modules and maps which lie in subgroups of appropriate algebraic K-groups.

3. Nonetheless, there is the spectacular work of Farrell and Jones (see [FJ1]) which naturally led to and verified variant versions of this conjecture for discrete subgroups of linear Lie groups. The main difficulty in mimicking these methods seems to be the very strong transfer formulae in topology. The "gamma element" is precisely a transfer-projection element.

   Philosophically speaking, the strong rigidity of the signature operator in families (a consequence of Hodge theory) makes its study more particular. Besides the difficulty in transferring the ideas of Farrell and Jones to index theory, this issue also arises in trying to develop a stratified index theory for operators on stratified spaces, parallel to [We4]. It would be interesting to see geometric examples of this phenomenon, for example, for some version of positive scalar curvature metrics on spaces with certain singularities. Of course, for particular classes of stratified spaces, and operators, one should be able to obtain such theories. See, for example, [Hi] for a theory of operators on $\mathbb{Z}/k$-manifolds.

4. In the past year, a number of counterexamples to versions of Baum-Connes, in particular the coarse version, were obtained using metric spaces that contain expanding graphs (see, for example, [HLS]). On the other hand, Kevin Whyte and the second author showed that some of these examples do not give counterexamples to the bounded version of the Borel conjecture. If, as seems likely, the isomorphism conjecture (or "stratified Borel conjecture" with rational coefficients) will be verified for hyperbolic groups, then the limit constructions of Gromov will also not lead to counterexamples to the topological versions of these problems.

5. Finally, it seems important to mention the circle of mathematical connections between index theory, cyclic homology, pseudoisotopy theory (=algebraic K-theory of spaces), and Goodwillie's calculus of functors. The Goodwillie idea (see, for example, [Go1, GW]) gives a powerful method for analyzing situations where assembly maps are not isomorphisms.

   The Borel conjecture are about the "linear part" of classification of manifolds. There are higher order "nonlinear terms" which are responsible for the counterexamples to the equivariant form. One should connect these ideas to families of operators and the work of Bismut and Lott [BL]. In addition, it would be good to have a better understanding of an index-theoretic (or perhaps we should say operator-algebraic) viewpoint on spectral invariants like Ray-Singer torsion and of eta invariants (and their higher versions, see [Lo1,2, We2,3 LP1, LL]).

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