COARSE OBSTRUCTIONS TO POSITIVE SCALAR CURVATURE IN NONCOMPACT ARITHMETIC MANIFOLDS

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ABSTRACT. Block and Weinberger show that an arithmetic manifold can be endowed with a positive scalar curvature metric if and only if its $\mathbb{Q}$-rank exceeds 2. We show in this article that these metrics are never in the same coarse class as the natural metric inherited from the base Lie group. Furthering the coarse C^*-algebraic methods of Roe, we find a nonzero Dirac obstruction in the K-theory of a particular operator algebra which encodes information about the quasi-isometry type of the manifold as well as its local geometry.

I. Introduction

In the course of showing that no manifold of non-positive sectional curvature can be endowed with a metric of positive scalar curvature, Gromov and Lawson [9] were led to consider what we would now call restrictions on the coarse equivalence type of complete noncompact manifolds of such positively curved metrics. In particular, they showed that such metrics cannot exist in manifolds for which there exists a degree one proper Lipschitz map from the universal cover to $\mathbb{R}^n$, now understood to be essentially a coarse condition. Block and Weinberger [2] investigate the situation in which no coarse conditions are imposed upon the complete metric, focusing on quotients $\Gamma\backslash G/K$ of symmetric spaces associated to a lattice Γ in an irreducible semisimple Lie group G. They show that the space $M = \Gamma\backslash G/K$ can be given a complete metric of uniformly positive scalar curvature $\kappa \geq \varepsilon > 0$ if and only if Γ is an arithmetic group of $\mathbb{Q}$-rank exceeding 2.

Note that the theorem of Gromov and Lawson [9] mentioned above establishes this theorem in the case of $\text{rank}_{\mathbb{Q}}\Gamma = 0$. In the higher rank cases, for which the resulting quotient space is noncompact, the metrics constructed by Block and Weinberger are however wildly different in the large when compared to the natural one on M inherited from the base Lie group G. In fact, their examples are all coarse quasi-isometric to rays. Their theory evokes a natural question: Can the metric be chosen so that it is simultaneously uniformly positively curved and coarsely equivalent to the natural metric induced by G?

One of the important developments in analyzing positive scalar curvature in the context of noncompact manifolds, especially when restricted to the coarse quasi-isometry type, is introduced by Roe [15], [14], who considers a higher index, analogous to the Novikov higher signature, that lives naturally in the K-theory of the C^*-algebra $C^*(M)$ of operators on M with finite propagation speed. He describes a map from the K-theory group $K_*(C^*(M))$ to the K-homology $K_*(\nu M)$ of the Higson corona space which admits a dual transgression map $H^*(\nu M) \to HX^*(M)$. If the Dirac operator on M is invertible, then the image of its index in $K_*(\nu M)$ vanishes, leading to vanishing theorems for the index paired with coarse classes from the transgression of νM. Roe's construction is used to show that a metric on a noncompact manifold cannot be uniformly positively curved if the Higson corona of the manifold contains an essential $(n-1)$-sphere. Such spaces are called *ultraspherical manifolds*.

The usual Roe algebra, however, is unsuited to provide information about the existence of positive scalar curvature metrics that exist on arithmetic manifolds, in particular because the corona is too anemic. For example, the space at infinity of a product of punctured two-dimensional tori is a simplex and therefore contractible. As a coarse object, the K-theory of the Roe algebra associated to this multi-product space can be identified with $K_*(C^*(\mathbb{R}^n_{\geq 0}))$. Yet Higson, Roe and Yu [11] have shown that the Euclidean cone cP on a single simplex P must satisfy $K_*(C^*(cP)) = 0$. Since the Euclidean hyperoctant $\mathbb{R}^n_{\geq 0}$ is simply the cone on an $(n-1)$-simplex, we find that $K_*(C^*(\mathbb{R}^n_{\geq 0}))$ is the trivial group and hence no obstructions are detectable. Even by considering the fundamental group of the manifold by tensoring the Roe algebra with $C^*\pi_1(M)$ we find this detection process unfruitful, since the K-theory group $K_*(C^*(M) \otimes C^*\pi_1(M))$ vanishes as well. What seems to be critical is how different elements of the fundamental group at infinity can be localized to different parts of the space at infinity.

In this article, we shall provide coarse indicial obstructions in the following noncompact manifolds: a finite product of punctured two-dimensional tori, a finite product of hyperbolic manifolds, the double quotient space $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ of unit volume tori, and more generally the double quotient space $\Gamma\backslash G/K$, where G is an irreducible semisimple Lie group, K its maximal compact subgroup and Γ an arithmetic subgroup of G. Note that the first two do not correspond to irreducible quotients, but an analysis of these spaces gives us the proper insight to attack the more general cases. A further research project will analyze this problem without the irreducibility assumption. The key feature in these particular manifolds M is that they contain hypersurfaces V that are coarsely equivalent to a product $E \times U$ of Euclidean space E with some iterated circle bundle U (i.e. a torus, Heisenberg group, or more generally a group of unipotent matrices). Moreover such a hypersurface decomposes the manifold M into a *coarsely excisive pair* (A, B) for which $A \cup B = M$ and $A \cap B = V$. A generalized form of the Mayer-Vietoris sequence constructed by Higson, Roe and Yu [11] provides the following:

$$\cdots \longrightarrow K_*(C^*_G(A)) \oplus K_*(C^*_G(B)) \longrightarrow K_*(C^*_G(M)) \longrightarrow K_{*-1}(C^*_G(V)) \longrightarrow \cdots$$

The boundary map $\partial : K_*(C^*_G(M)) \longrightarrow K_{*-1}(C^*_G(E \times U))$ sends $\mathrm{Ind}_M(D)$, the index of the spinor Dirac bundle on the universal cover lifted from that on M, to $\mathrm{Ind}_{E\times U}(D)$. To see that these indices are indeed nonzero, we note that there is a boundary map $K_{*-1}(C^*_G(E \times U)) \to K_{*-\dim E}(C^*_G(\mathbb{R} \times U))$, which sends index to index. We show that the index of the Dirac operator in the latter group, however, is nonzero by noting that the Gromov-Lawson-Rosenberg conjecture is true for nilpotent groups and hence provides an appropriate nonzero obstruction.

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II. The Generalized Roe Algebra

The *coarse category* is defined to contain metric spaces as its objects and maps $f : (X, d_X) \to (Y, d_Y)$ between metric spaces as its morphisms satisfying the following expansion and properness conditions: (a) for each $R > 0$ there is a corresponding $S > 0$ such that, if $d_X(x_1, x_2) \leq R$ in X, then $d_Y(f(x_1), f(x_2)) \leq S$, (b) the inverse image $f^{-1}(B)$ under f of each bounded set $B \subseteq Y$ is also bounded in X. Such a function will be designated a *coarse map*, and two coarse maps $f, g : X \to Y$ are said to be *coarsely equivalent* if their mutual distance of separation $d_Y(f(x), g(x))$ is uniformly bounded in x. Naturally two metric spaces are coarsely equivalent if there exist maps from one to the other whose compositions are coarsely equivalent to the appropriate identity maps. Two metrics g_1 and g_2 on the same space M are said to be coarsely equivalent if (M, g_1) and (M, g_2) are coarsely equivalent metric spaces.

Following Roe [14], we recall a Hilbert space H is an M-module for a manifold M if there is a representation of $C_0(M)$ on H, that is, a C^*-homomorphism $C_0(M) \to B(H)$. We will say that an operator $T : H \to H$ is *locally compact* if, for all $\varphi \in C_0(M)$, the operators $T\varphi$ and φT are compact on H. We define the *support* of φ in an M-module H to be the smallest closed set $K \subseteq M$ such that, if $f \in C_0(M)$ and $f\varphi \neq 0$, then $f|_K$ is not identically zero. Consider the $\widetilde{M}$-module $H = L^2(\widetilde{M})$, where $\widetilde{M}$ is the universal cover of M endowed with the appropriate metric lifted from the base space. Let $\pi : \widetilde{M} \to M$ be the usual projection map and for any $\varphi, \psi \in C_0(\widetilde{M})$ consider the collection $\Gamma(\varphi, \psi)$ of paths $\gamma : [0, 1] \to \widetilde{M}$ in $\widetilde{M}$ originating in Supp (φ) and ending in Supp (ψ). Denote by $L[\,\gamma\,]$, for $\gamma \in \Gamma(\varphi, \psi)$, the maximum distance of any two points on the projection of the curve γ in M by π, i.e. $L[\,\gamma\,] = \sup_{x,y \in [0,1]} d(\pi \circ \gamma(x), \pi \circ \gamma(y))$.

Definition: Let M be a manifold with universal cover $\widetilde{M}$. We say that an operator T on $L^2(\widetilde{M})$ has *generalized finite propagation* if there is a constant $R > 0$ such that $\varphi T \psi$ is identically zero in $B(H)$ whenever $\varphi, \psi \in C_0(\widetilde{M})$ satisfies

$$\inf_{\gamma \in \Gamma(\varphi,\psi)} L[\,\gamma\,] > R.$$

The infimum of all such R will be the *generalized propagation speed* of the operator T. If $G = \pi_1(M)$ is the fundamental group of M, we denote by $D^*_G(M)$ to be the norm closure of the C^*-algebra of all locally compact, G-equivariant, generalized finite propagation operators on H.

Let M be a manifold and $\widetilde{M}$ its universal cover. Let $T : H \to H$ be an operator on $H = L^2(\widetilde{M})$. Consider the subset $Q \subseteq \widetilde{M} \times \widetilde{M}$ of pairs (m, m') for which there exist functions $\varphi, \psi \in C_0(\widetilde{M})$ such that $\varphi(m) \neq 0$, $\psi(m') \neq 0$ and $\varphi T \psi$ does not identically vanish. We will say that the *support* of T is the complement in $\widetilde{M} \times \widetilde{M}$ of Q. For such two points $m, m' \in \widetilde{M}$, let $\gamma_{mm'} : [\,0, 1\,] \to \widetilde{M}$ be the path of least length joining m and m' in $\widetilde{M}$. We consider the projection of this path into M by π and take the greatest distance between two points on this projected path. Then it is easy to see that an operator T has generalized finite propagation, as previously defined, if

$$\sup_{m,m'} \sup_{x,y \in [0,1]} d(\pi \circ \gamma_{mm'}(x), \pi \circ \gamma_{mm'}(y)) < \infty.$$

Definition: Consider the norm closure I of the ideal in $D^*_G(M)$ generated by operators T whose matrix representation, parametrized by $\widetilde{M} \times \widetilde{M}$, satisfies the condition that $(\pi \times \pi)(\mathrm{Supp}\, T)$ is bounded in $M \times M$. Then the *generalized Roe algebra*, denoted by $C^*_G(M)$, is obtained as the quotient $D^*_G(M)/I$. Two operators in $D^*_G(M)$ belong to the same class in $C^*_G(M)$ if their nonzero entries differ on at most a bounded set when viewed from the perspective of the base space.

Examples:

(1) Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be operator on L^2-functions on the real line given by $(Tg)(x) = g(x+1)$ for all $g \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$. Then for any $\varphi, \psi \in C_0(\mathbb{R})$, $(\varphi T \psi)g(x) = \varphi(x+1)g(x+1)\psi(x)$. If φ is supported at $m = 1$ and ψ is supported at $m' = 0$, then $(\varphi T\psi)g$ is nonzero for any g supported at $x = 1$. Hence $(0, 1) \in \operatorname{Supp} T$. It is easy to see that $(m, m') \in \operatorname{Supp} T$ if and only if $m' - m = 1$. The propagation speed of T is 1. If we write T as a matrix parametrized by $\mathbb{R} \times \mathbb{R}$, all the nonzero entries will lie at distance one from the diagonal.

(2) Let M be the cylinder $S^1 \times \mathbb{R}$ with its universal cover $\widetilde{M} = \mathbb{R}^2$. An operator in the algebra $D^*_G(M)$ will be some $T : H \to H$ on $L^2(\mathbb{R}^2)$, which is of finite propagation speed (in the usual sense) in the direction projecting down to the noncompact direction in M, but has no such condition in the orthogonal direction corresponding to the compact direction of M. In this direction, however, the operator is controlled by the condition that it be $\mathbb{Z}$-equivariant. It is apparent that the operator, when restricted to individual fibers, has finite propagation speed, although there is no requirement that the speed to be uniformly bounded across all fibers.

(3) Let $M = \overset{\circ}{\mathbb{RP}}{}^n$, $n \geq 3$, the once-punctured real projective space, expressible as the quotient $(S^{n-1} \times \mathbb{R})/\mathbb{Z}_2$. Certainly M is coarsely equivalent to the ray $[\,0, \infty)$ and is covered by the space $\widetilde{M} = S^{n-1} \times \mathbb{R}$, where the points (s, r) and $(-s, -r)$ are identified by the projection map to M. Let $T : L^2(\widetilde{M}) \to L^2(\widetilde{M})$ be given by the reflection $(Tf)(s, r) = f(s, -r)$. Consider $\varphi_i, \psi_i \in C_0(\widetilde{M})$ compactly supported on $S^{n-1} \times [\,-i-1, -i\,]$ and $S^{n-1} \times [\,i, i+1\,]$, respectively. Notice that $\varphi T \psi$ will never be identically zero, and yet the length $L_i[\,\gamma\,]$ associated to φ_i and ψ_i will always be at least i. Hence the operator T is *not* of generalized finite propagation speed and therefore not an element of the generalized Roe algebra $C^*_G(M)$.

The notion of a generalized elliptic operator is available in [14], [15] and [16], but we include its definition here for completeness.

Definition: Let M be a space and let H be an M-module. If D is an unbounded self-adjoint operator on H, then we say that D is a *generalized elliptic operator* on H if

(a) there is a constant $c > 0$ such that, for all $t \in \mathbb{R}$, the unitary operator e^{itA} has bounded propagation on H and and its propagation bound is less than $c|t|$, and
(b) there is $n > 0$ such that $(1 + D^2)^{-n}$ is locally traceable.

Lemma 1: Let D a generalized elliptic operator in $L^2(M, S)$. Suppose that $\widetilde{D}$ is the lifted operator on $\widetilde{M}$. If $\Phi : \mathbb{R} \to \mathbb{R}$ is compactly supported, then $\Phi(\widetilde{D})$ lies in the generalized Roe algebra $C^*_G(M)$.

Proof: (cf. [14], [6]) Suppose that Φ has compactly supported Fourier transform and denote by $\widehat{\Phi}$ the Fourier transform of Φ. We may write

$$\Phi(\widetilde{D}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{\Phi}(t)\, e^{it\widetilde{D}}\, dt.$$

It is known that $e^{it\widetilde{D}}$ has finite propagation speed, and since $\widehat{\Phi}$ is compactly supported, the integral is defined and has a generalized propagation bound. Moreover, by construction $\widetilde{D}$ is $\pi_1(M)$-equivariant. So $\Phi(\widetilde{D})$ is $\pi_1(M)$-equivariant as well. Therefore if $\widehat{\Phi}$ is compactly supported, then $\Phi(\widetilde{D})$ lies in $D^*_G(M)$ and passes to an element of the quotient $C^*_G(M)$. However, functions with compactly supported Fourier transform form a dense set in $C_0(\mathbb{R})$ and the functional calculus map $f \mapsto f(\widetilde{D})$ is continuous, so the result holds for all $\Phi \in C_0(\mathbb{R})$. $\square$

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a *chopping function* on $\mathbb{R}$, i.e. an odd continuous function with the property that $\chi(x) \to \pm 1$ as $x \to \pm\infty$. In addition, denote by $B^*_G(M)$ the *multiplier algebra* of $C^*_G(M)$, that is, the collection of all operators S such that ST and TS belong to $C^*_G(M)$ for all $T \in C^*_G(M)$. Then $B^*_G(M)$ contains $C^*_G(M)$ as an ideal. If D is a generalized elliptic operator on M and $\widetilde{D}$ its lift to $\widetilde{M}$, then $\chi(\widetilde{D})$ belongs to $B^*_G(M)$. In addition, since $\chi^2 - 1 \in C_0(\mathbb{R})$, we have $\chi(\widetilde{D})^2 - 1 \in C^*_G(M)$. Moreover, since the $\mathbb{Z}_2$-grading renders the decompositions

$$\chi(\widetilde{D}) = \begin{pmatrix} 0 & \chi(\widetilde{D})_- \\ \chi(\widetilde{D})_+ & 0 \end{pmatrix}, \; \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

it follows that $\varepsilon\,\chi(\widetilde{D}) + \chi(\widetilde{D})\,\varepsilon = 0$. By the discussion in [14], it follows that $F = \chi(\widetilde{D})$ is a Fredholm operator and admits an index $\operatorname{Ind} F \in K_0(C^*_G(M))$. In addition, any two chopping functions χ_1 and χ_2 differ by an element of $C_0(\mathbb{R})$. By the lemma above, we have $\chi_1(\widetilde{D}) - \chi_2(\widetilde{D}) \in C^*_G(M)$, so they define the same elements of K-theory. The common value for $\operatorname{Ind} F$ is denoted $\operatorname{Ind}(D)$ and called the *generalized coarse index* of D. We write $C^*_G(M)$ and $\operatorname{Ind}(D)$ instead of $C^*_G(\widetilde{M})$ and $\operatorname{Ind}(\widetilde{D})$ to indicate that the construction is initiated by a generalized Dirac operator on the base space. The following statements are standard results of index theory; one may consult [14] and [15] for the essentially identical proof in the nonequivariant case.

Proposition 1: Let D be a generalized elliptic operator in $L^2(M,S)$. If 0 does not belong to the spectrum of $\widetilde{D}$, then the generalized coarse index $\operatorname{Ind} D$ vanishes in $K_0(C^*_G(M))$.

Proposition 2: Let $\widetilde{D}$ the lift of a generalized elliptic operator in $L^2(\widetilde{M},S)$. In the ungraded case, if there is a gap in the spectrum of $\widetilde{D}$, then the index $\operatorname{Ind} D$ vanishes in $K_1(C^*_G(M))$.

Corollary: Let M be a complete spin manifold. If M has a metric of uniformly positive scalar curvature in some coarse class, then the generalized coarse index of the spinor Dirac operator vanishes.

We now embark on the task of computing the K-theory of this algebra and of coarse indices. Let (M,d) be a proper metric space. For any subset $U \subset M$ and $R > 0$, we denote by $\mathrm{Pen}(U,R)$ the open neighborhood of U consisting of points $x \in M$ for which $d(x,U) < R$. Let A and B be closed subspaces of M with $M = A \cup B$. We then say that the decomposition (A,B) is a *coarsely excisive pair* if for each $R > 0$ there is an $S > 0$ such that

$$\mathrm{Pen}(A,R) \cap \mathrm{Pen}(B,R) \subseteq \mathrm{Pen}(A \cap B, S).$$

We wish to analyze this decomposition in the following context.

Given general C^*-algebras $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$ for which $\mathcal{M} = \mathcal{A} + \mathcal{B}$, we have the Mayer-Vietoris sequence

$$\cdots \longrightarrow K_{j+1}(\,\mathcal{M}\,) \longrightarrow K_j(\,\mathcal{A} \cap \mathcal{B}\,) \longrightarrow K_j(\,\mathcal{A}\,) \oplus K_j(\,\mathcal{B}\,) \longrightarrow K_j(\,\mathcal{M}\,) \longrightarrow \cdots$$

The standard proof for the existence of such a sequence is developed from the isomorphism $K_*(\mathcal{T}) \cong K_{*-1}(\mathcal{M})$, where $\mathcal{T}$ is the suspension of $\mathcal{M}$. A short discussion of this construction is given in [11]. We are in particular interested in exploiting the boundary map $\partial : K_j(\,\mathcal{M}\,) \to$

$K_{j-1}(\mathcal{A}\cap\mathcal{B})$ to transfer information about the index of the Dirac operator on a complete noncompact manifold M to information about that on some hypersurface V. For our purposes, we wish to set $\mathcal{M}$ to be the generalized Roe algebra $C^*_G(M)$ on M, while $\mathcal{A}$ and $\mathcal{B}$ represent analogous operator algebras on closed subsets A and B, where (A, B) form a coarsely excisive decomposition of M. To construct the boundary map in question, we require a few technical lemmas and notion of equivariant operators with generalized finite propagation on a subset of M. The proof of the first lemma follows the same argument as that in [11] and is stated without proof.

Definition: Let A be a closed subspace of a proper metric space M. Denote by $D^*_G(A, M)$ the C^*-algebra of all operators T in $D^*_G(M)$ such that $\operatorname{Supp} T \subseteq \text{Pen}(\pi^{-1}(A), R) \times \text{Pen}(\pi^{-1}(A), R)$, for some $R > 0$. Let $C^*_G(A, M)$ be the quotient $D^*_G(A, M)/I$.

Lemma 2: Let (A, B) be a decomposition of M. Then

(1) $C^*_G(A, M) + C^*_G(B, M) = C^*_G(M)$.
(2) $C^*_G(A, M)\cap C^*_G(B, M) = C^*_G(A\cap B, M)$ if in addition we assume that (A, B) is coarsely excisive.

Lemma 3: Suppose that the inclusion $V \subset M$ induces an injection $\pi_1(V) \to \pi_1(M)$ on the level of fundamental groups. There is an isomorphism $K_*(C^*_G(V)) \cong K_*(C^*_G(V, M))$.

Proof: Let $\pi : \widetilde{M} \to M$ be the projection map. Consider the C^*-algebra $C^*(\text{Pen}(\pi^{-1}(V), n), \pi_1(M))$ given by the quotient by I of the C^*-algebra of locally compact, $\pi_1(M)$-equivariant operators on the n-neighborhood penumbra $\text{Pen}(\pi^{-1}(V), n)$. Then

$$C^*_G(V, M) = \varinjlim C^*(\text{Pen}(\pi^{-1}(V), n), \pi_1(M)).$$

The inclusion map $i : \pi^{-1}(V) \hookrightarrow \text{Pen}(\pi^{-1}(V), n)$ is a coarse equivalence. Since by the construction the generalized Roe algebra its operators are defined up to their bounded parts, the map i induces a series of isomorphisms

$$\begin{aligned} K_*(C^*(\pi^{-1}(V), \pi_1(M))) &\cong K_*(C^*(\text{Pen}(\pi^{-1}(V), n), \pi_1(M)) \\ &\cong K_*(\lim C^*(\text{Pen}(\pi^{-1}(V), n), \pi_1(M)) \\ &\cong K_*(C^*_G(V, M)). \end{aligned}$$

Since $\pi_1(V) \hookrightarrow \pi_1(M)$ is an injection, the inverse image $\pi^{-1}(V) \subseteq \widetilde{M}$ is a disjoint union of isomorphic copies of $\widetilde{V}$, parametrized by the coset space $\pi_1(M)/\pi_1(V)$. Therefore, there is a one-to-one correspondence between $\pi_1(M)$-equivariant operators on $\pi^{-1}(V)$ and $\pi_1(V)$-equivariant operators on $\widetilde{V}$. Hence $C^*(\pi^{-1}(V), \pi_1(M)) \cong C^*_G(V)$. We then have $K_*(C^*_G(V)) \cong K_*(C^*_G(V, M))$, as desired. $\square$

Let (A, B) be a coarsely excisive decomposition of M such that $V = A \cap B$ satisfies $\pi_1(V) \hookrightarrow \pi_1(M)$. The boundary operator $\partial : K_j(C^*_G(A, M) + C^*_G(B, M)) \longrightarrow K_{j-1}(C^*_G(A, M) \cap C^*_G(B, M))$ arising from the coarse Mayer-Vietoris sequence is by the previous lemmas truly a map

$$\partial : K_*(C^*_G(M)) \to K_{*-1}(C^*_G(V)).$$

Theorem: (Boundary of Dirac is Dirac) Consider a coarsely excisive decomposition (A, B) of M and let $V = A \cap B$. If $\partial : K_*(C^*_G(M)) \to K_{*-1}(C^*_G(V))$ is the boundary map from the Mayer-Vietoris sequence derived above, then we have $\partial\,(\text{Ind}_M(D)) = \text{Ind}_V(D)$.

Remark: Here $\mathrm{Ind}_M(D)$ and $\mathrm{Ind}_V(D)$ represent the generalized coarse indices of the spinor Dirac operators on M and V, respectively. We will continue to use a subscript if the space to which the index is related is ambiguous. The "boundary of Dirac is Dirac" principle is essentially equivalent to Bott periodicity in topological K-theory. In all cases considered here, there are commutative diagrams relating topological boundary to the boundary operator arising in the K-theory of C^*-algebras, and, on the topological side, a consideration of symbols suffices. See [16], [10], [15] and [21].

Theorem 1: The n-fold product M of punctured two-dimensional tori does not have a metric of uniform positive scalar curvature in the same coarse equivalence class as the positive hyperoctant with its standard Euclidean metric.

Proof: Consider the projection map $p : M \to \mathbb{R}^n_{\geq 0}$ from the product $M = \overset{\circ}{T} \times \cdots \times \overset{\circ}{T}$ to the positive hyperoctant, where each component p_i is the quasi-isometric projection of the punctured torus onto the positive reals numbers. Take a hypersurface $S \subset \mathbb{R}^n_{\geq 0}$ sufficiently far from the origin so that the inverse image of every point on S is an n-torus, and so the space V is coarsely equivalent to the $(2n-1)$-dimensional noncompact manifold $\mathbb{R}^{n-1} \times T^n$. The complement of the hypersurface V consists of two noncompact components. Define A to be the closure of the component containing the inverse image $p^{-1}(\mathbf{0})$ of the origin in $\mathbb{R}^n_{\geq 0}$. Take B the closure of $M \backslash A^\circ$. Then the pair (A, B) forms a coarsely excisive decomposition of the space M whose intersection is $A \cap B = V$.

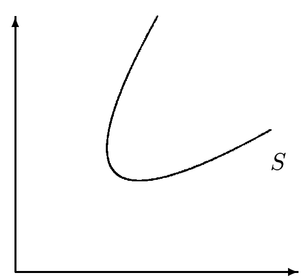

FIGURE 1. The hypersurface S in $\mathbb{R}^n_{\geq 0}$.

Consider the generalized coarse index $\mathrm{Ind}_M(D) \in K_*(C^*_G(M))$ of the lifted classical Dirac operator on the pullback spinor bundle of the universal cover $\widetilde{M}$. Note that $\pi_1(M)$ is the n-fold product $F_2 \times \cdots \times F_2$ of free groups, and that $\pi_1(V) \cong \pi_1(\mathbb{R}^{n-1} \times T^n) \cong \mathbb{Z}^n$. Hence there is an injection $\pi_1(V) \hookrightarrow \pi_1(M)$ and the K-theoretic Mayer-Vietoris sequence applies. The boundary map ∂ of this sequence satisfies $\partial\,(\,\mathrm{Ind}_M(D)) = \mathrm{Ind}_V(D) \in K_*(C^*_G(V))$. However, V is coarsely equivalent to the hypersurface $\mathbb{R}^{n-1} \times T^n$, so the index $\mathrm{Ind}_V(D)$ can be taken to live in $K_{*-1}(C^*_G(\mathbb{R}^{n-1} \times T^n))$. Note that n will be taken to be at least 2. There is yet another boundary map $K_{*-1}(C^*_G(\mathbb{R}^{n-1} \times T^n)) \to K_{*-n+1}(C^*_G(\mathbb{R} \times T^n))$ by peeling off $n-2$ copies of the real line. This boundary map (or composition of $n-2$ boundary maps) preserves index.

Recall that $D^*_G(M)$ is the norm closure of the C^*-algebra of all locally compact, $\pi_1(M)$-equivariant, generalized finite propagation operators on $L^2(\widetilde{M})$, and $I \subseteq D^*_G(M)$ is the closure of the ideal of such operators T that satisfies the condition that $(\pi \times \pi)(\operatorname{Supp} T)$ is bounded in $M \times M$. The short exact sequence $0 \longrightarrow I \longrightarrow D^*_G(\mathbb{R} \times T^n) \longrightarrow C^*_G(\mathbb{R} \times T^n) \longrightarrow 0$ gives rise to the six-term exact sequence in K-theory:

$$\begin{tikzcd}
K_0(I) \arrow[r] & K_0(D^*_G(\mathbb{R} \times T^n)) \arrow[r] & K_0(C^*_G(\mathbb{R} \times T^n)) \arrow[d] \\
K_1(C^*_G(\mathbb{R} \times T^n)) \arrow[u] & K_1(D^*_G(\mathbb{R} \times T^n)) \arrow[l] & K_1(I) \arrow[l]
\end{tikzcd}$$

Notice that the map $K_*(I) \to K_*(D^*_G(\mathbb{R} \times T^n))$ induced by the inclusion is the zero map by an Eilenberg swindle argument. Hence both maps $K_*(D^*_G(\mathbb{R} \times T^n)) \to K_*(C^*_G(\mathbb{R} \times T^n))$ are injections.

If n is even, the generalized coarse index $\operatorname{Ind}_{\mathbb{R} \times T^n}(D)$ of the Dirac operator D resides in $K_1(D^*_G(\mathbb{R} \times T^n))$. Certainly the image of this index under the boundary map $K_1(D^*_G(\mathbb{R} \times T^n)) \to K_0(D^*_G(T^n))$ is the index $\operatorname{Ind}_{T^n}(D)$ of D on the n-torus. Since T^n does not have a metric of positive scalar curvature at all, the obstruction $\alpha(T^n, f) \in K_*(C^*(\mathbb{Z}^n))$, where $f : T^n \to B\mathbb{Z}^n$ is the classifying map, is nonvanishing. This "index" is constructed by Rosenberg in [17]. This special case of the Gromov-Lawson-Rosenberg conjecture holds for the group $\mathbb{Z}^n$ [5]. This index maps to our generalized coarse index $\operatorname{Ind}_{T^n} D$ under the isomorphism $K_*(C^*(\mathbb{Z}^n)) \cong K_*(D^*_G(T^n))$. Hence the index of D in $K_1(D^*_G(\mathbb{R} \times T^n))$ is nonzero, and its projection onto the group $K_1(C^*_G(\mathbb{R} \times T^n))$ is nonzero as well. This argument gives us the necessary index obstruction.

If n is odd, we apply the same argument as above with respect to the map $K_0(D^*_G(\mathbb{R} \times T^n)) \to K_0(C^*_G(\mathbb{R} \times T^n))$. □

The extension of this method to multifold products of hyperbolic manifolds involves the Margulis lemma, which states that in such a space there exists a small positive constant $\mu = \mu_n$ such that the subgroup $\Gamma_\mu(V, v) \subseteq \pi_1(V, v)$ generated by loops of length less than or equal to μ based at $v \in V$ is almost nilpotent, i.e. it contains a nilpotent subgroup of finite index. It can be shown that there exist such cusps, or submanifolds $C \subset V$ with compact convex boundary containing v, such that C is diffeomorphic to the product $\partial C \times \mathbb{R}_+$, where ∂C is diffeomorphic to an $(n-1)$-dimensional nilmanifold with fundamental group containing $\Gamma_\mu(V, v)$. Here a nilmanifold signifies a quotient N/Γ of a nilpotent Lie group by a cocompact lattice Γ. The nilmanifolds that arise in this context as boundaries of pseudospheres will have a naturally flat structure.

Theorem 2: An n-fold product of hyperbolic manifolds has no uniform positive scalar curvature metric coarsely equivalent to the usual Euclidean metric on the positive Euclidean hyperoctant.

Proof: Without loss of generality, it suffices to consider the case in which the noncompact hyperbolic spaces have only one cusp. Let m be the dimension of this product manifold. As in the multifold product of tori, there is a positive $b \in \mathbb{R}$ such that on each hyperbolic space $\mathcal{H}_i$ the inverse image of each point $x \geq b$ under the projection $\mathcal{H}_i \to \mathbb{R}_{\geq 0}$ is by Margulis' lemma a flat compact connected Riemannian manifold of finite dimension. Consider the inverse image $\mathcal{V}$ under the induced product map $p : \mathcal{H}_1 \times \cdots \times \mathcal{H}_m \to \mathbb{R}^m_{\geq 0}$ of the same hypersurface as described in the previous theorem. By Bieberbach's theorem, every flat compact connected Riemannian manifold admits a normal Riemannian covering by a flat torus of the same dimension. Hence $\mathcal{V}$ is covered by some product of Euclidean space and a higher-dimensional torus. Any metric of positive scalar curvature on $\mathcal{V}$ would certainly lift to such a metric in this covering space. Using the same induction argument as before, we show that such a metric is obstructed by the presence of a nonzero Dirac class. $\square$

III. Noncompact Quotients of Symmetric Spaces: A Special Case

We wish to apply the above techniques to the irreducible case $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$. This space is not locally symmetric because $\mathrm{SL}_n(\mathbb{Z})$ does not act freely on $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$, so the quotient is not a Riemannian manifold. Let X^* be any finite-sheeted branched cover of the double quotient $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$, i.e. a manifold corresponding to a subgroup of finite index in $\mathrm{SL}_n(\mathbb{Z})$.

The Iwasawa decomposition gives a unique way of expressing the group $\mathrm{SL}_n(\mathbb{R})$ as a product $\mathrm{SL}_n(\mathbb{R}) = NAK$, where N is the subgroup of standard unipotent matrices (upper triangular matrices with all diagonal entries equal to 1), A the subgroup of $\mathrm{SL}_n(\mathbb{R})$ consisting of diagonal matrices with positive entries, and K the orthogonal subgroup $\mathrm{SO}_n(\mathbb{R})$. The quotient $X \equiv \mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ can then be trivially seen to have $\frac{n(n-1)}{2}$ compact directions arising from N, an additional $n-1$ noncompact directions from A, and an $(n-2)$-dimensional simplex as its boundary. In short, the bordified space $\overline{X}$ is coarsely an $(n-1)$-simplex.

Theorem 3: Let $n \geq 3$ and let $\mathrm{SL}_n(\mathbb{Z})^*$ be a torsion-free subgroup of $\mathrm{SL}_n(\mathbb{Z})$ of finite index. Then the manifold $X^* = \mathrm{SL}_n(\mathbb{Z})^*\backslash\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ lacks a uniform positive scalar curvature metric that is coarsely equivalent to the natural one inherited from $\mathrm{SL}_n(\mathbb{R})$.

Proof: To build the appropriate hypersurface in X^*, consider first the orbifold X, which by the above discussion can be expressed as $\Gamma\backslash NAK/K$, where $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ and $K = \mathrm{SO}_n(\mathbb{R})$. Consider the Weyl chamber corresponding to the subset $A^+ \subset A$ of positive diagonal matrices with decreasing entries, i.e.

$$A^+ = \left\{ \begin{pmatrix} e^{a_1} & 0 & \dots & 0 \\ 0 & e^{a_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{a_n} \end{pmatrix} : a_1 > a_2 > \dots > a_{n-1} > a_n \right\}.$$

Here $a_n = -(a_1 + a_2 + \cdots + a_{n-1})$. The coordinates $a_1, a_2, \dots, a_{n-1}$ parametrize the $n-1$ noncompact directions of X.

The closure of this Weyl chamber corresponding to $a_1 > a_2 > \cdots > a_n$ is a simplex with one boundary face at infinity. One can construct a closed, convex subset $A_H^+ \subset A^+$ with the following properties (see Figure 1): the set A_H^+ is bounded away from the boundary faces $a_i = a_{i+1}$ and the quotient $\mathcal{F} = \Gamma\backslash NA_H^+K/K$ of the resulting Siegel set NA_H^+K has a boundary $W \equiv \partial\mathcal{F}$ which is coarsely equivalent to an iterated circle bundle over $(n-2)$-dimensional Euclidean space. More precisely, there is a coarse equivalence $W \to \mathbb{R}^{n-2}$ such that the fiber over each point is a compact arithmetic quotient of the group of unipotent matrices. (Notice that in general arithmetic subgroups are not unipotent, but we create this hypersurface W for the explicit purpose of exploiting the unipotent parts of X.) The reader may wish to consider the $n = 3$ case and construct the subset

$A_H^+ \subset A^+$ given by

$$A_H^+ = \left\{ k(1, b, -1-b) : 0 \leq b \leq \frac{1}{2},\ k \geq L \right\},$$

where L is sufficiently large. Refer to the following section for a discussion about the action of Γ on W.

Since X^* is a finite-sheeted branch cover of X, there is a natural projection map $p : X^* \to X$. The set $p^{-1}(W) \subset X^*$ will be a disjoint union of copies of W. Let W^* be just one connected component. This noncompact space W^* partitions the space X^* into a coarsely excisive pair whose closures (Y, Z) satisfy the equalities $Y \cup Z = M^*$ and $Y \cap Z = W^*$. If $\mathrm{Ind}_{X^*}(D)$ denotes the generalized coarse index of the classical spinor Dirac operator on $\widetilde{X^*}$, then the Mayer-Vietoris map $\partial : K_*(C_G^*(X^*)) \to K_{*-1}(C_G^*(W^*))$ defined in the previous chapter satisfies $\partial(\mathrm{Ind}_{X^*}(D)) = \mathrm{Ind}_{W^*}(D) = \mathrm{Ind}_{\mathbb{R}^{n-2}\times U^m}(D)$, where U^m is the compact fiber of the iterated circle bundle of dimension $m = \frac{n(n-1)}{2}$. Applying the same argument as before, we need only to show that the index of the Dirac operator in $K_*(D_G^*(U^m))$ is nonzero. However, the compact space U^m is a quotient of a nilpotent group by a cocompact lattice, and hence by Gromov and Lawson [9] has no metric of positive scalar curvature at all. As with the theorem for punctured tori, there is a nonvanishing Rosenberg index $\alpha(U^m) \in K_*(C_r^*(\pi))$, where $\pi = \pi_1(U^m)$, which maps to the generalized coarse index in $K_*(D_G^*(U^m))$, as desired. Here the Gromov-Lawson-Rosenberg conjecture is true since U^m is a nilmanifold. $\square$

Remark: One can avoid the detailed construction of W by accepting the coarse simplicial picture of $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$. To define an appropriate hypersurface that stays a bounded distance from the simplicial faces, except for the face at infinity, one can merely take the open cone on an $(n-3)$-dimensional sphere in the interior of the boundary face, and close this cone under the action of $N/(N \cap \Gamma)$.

IV. The General Noncompact Arithmetic Case

Let $\mathbf{G}$ be an affine algebraic group defined over $\mathbb{Q}$. We say that $\mathbf{G}$ is *semisimple* if its radical (i.e. its greatest connected normal solvable subgroup) is trivial. For such a $\mathbf{G}$, we denote its real locus $\mathbf{G}(\mathbb{R})$ by G, which is a semisimple Lie group with finitely many connected components. It is well known that the spherical Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$ associated with $\mathbf{G}$ over $\mathbb{Q}$ is a connected infinite simplicial complex if $\mathrm{rank}_{\mathbb{Q}}(\mathbf{G}) > 1$. The simplices of $\Delta_{\mathbb{Q}}(\mathbf{G})$ correspond bijectively to the proper rational parabolic subgroups of $\mathbf{G}$. If $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$, then there are only finitely many Γ-conjugacy classes of rational parabolic subgroups, so the quotient $\Gamma\backslash\Delta_{\mathbb{Q}}(\mathbf{G})$ is a finite simplicial complex, called the *Tits complex* of $\Gamma\backslash G/K$ and denoted $\Delta(\Gamma\backslash G/K)$. Here K is a maximal compact subgroup of G. See [13] for details.

Such a real semisimple Lie group G has a decomposition PK where P is a parabolic subgroup of G and K is maximal compact. This parabolic P satisfies the relation $P = C_G(A)N$, where $A \subset P$ is a connected maximal split torus with centralizer $C_G(A)$ and N is the unipotent radical of P. The Langlands decomposition of a parabolic P gives $P = NAM$, where $MA = C_G(A)$, the quotient $M/Z(M)$ is semisimple and $Z(M)$ is compact. Of course this decomposition depends on P and the point $x_0 \in G$ fixed by K.

Recall that a subgroup Γ of G is an *arithmetic lattice* if there exist

(1) a closed subgroup G' of some $\mathrm{SL}_\ell(\mathbb{R})$ such that G' is defined over $\mathbb{Q}$,
(2) compact normal subgroups $K \leq G$ and $K' \leq G'$, and
(3) an isomorphism $\phi : G/K \to G'/K'$

such that $\phi(\overline{\Gamma})$ is commensurable with $\overline{G'_{\mathbb{Z}}}$, where $\overline{\Gamma}$ and $\overline{G'_{\mathbb{Z}}}$ are the images of Γ and $G'_{\mathbb{Z}}$ in G/K and G'/K', respectively.

The effect of an arithmetic lattice Γ on the components of the Langlands decomposition is as follows. Let $P = NAM$ be a minimal parabolic $\mathbb{Q}$-subgroup, and let T be a maximal $\mathbb{Q}$-split torus of G. Then M satisfies the equality $C_G(T) = TM$. Since T is maximal, the subgroup M contains no $\mathbb{Q}$-split tori. By definition, we have $\mathrm{rank}_{\mathbb{Q}} M_{\mathbb{Z}} = 0$. Hence, the arithmetic subgroups of M are cocompact in M. Since the intersection of $G_{\mathbb{Z}}$ with M is an arithmetic subgroup of M, every quotient of M by an arithmetic subgroup of G yields a compact quotient. A similar argument holds for the subgroup N of G.

To understand the coarse type of $\Gamma\backslash G/K$, we appeal to Ji and MacPherson [13] in their proof of a conjecture of Siegel. In particular, let $\mathbf{P}_0 = \mathbf{G}, \mathbf{P}_1, \dots, \mathbf{P}_n$ be representatives of the Γ conjugacy classes of rational parabolic subgroups of $\mathbf{G}$. For each i, let $\mathbf{P}_i = N_{\mathbf{P}_i} M_{\mathbf{P}_i} A_{\mathbf{P}_i}$ be the Langlands decomposition of $\mathbf{P}_i$. Then there exists bounded $\omega_i \subset N_{\mathbf{P}_i} M_{\mathbf{P}_i}$ and Siegel sets $\omega_i \times A_{\mathbf{P}_i,t} \subset N_{\mathbf{P}_i} M_{\mathbf{P}_i} \times A_{\mathbf{P}_i}$ such that

(1) each Siegel set $\omega_i \times A_{\mathbf{P}_i,t}$ is mapped injectively into $\Gamma\backslash G/K$;
(2) the image of ω_i in $(\Gamma \cap P_i)\backslash N_{\mathbf{P}_i} M_{\mathbf{P}_i}$ is compact;
(3) if we identify $\omega_i \times A_{\mathbf{P}_i,t}$ with its image in $\Gamma\backslash G/K$, then $\Gamma\backslash G/K$ can be decomposed in the following disjoint union

$$\Gamma\backslash G/K = \coprod_{i=0}^{n} \omega_i \times A_{\mathbf{P}_i,t}.$$

Here the subset $A_{\mathbf{P}_i,t} \equiv \{a \in A_{\mathbf{P}_i} : \alpha_i(\log a) > t,\ i = 1, \dots, r\}$ is a shift of the positive chamber $A^+_{\mathbf{P}_i} \equiv \{a \in A_{\mathbf{P}_i} : \alpha_i(\log a) > 0,\ i = 1, \dots, r\}$, where the α_i are the associated set of simple roots and t is sufficiently large. Using this so-called precise reduction theory and identifying $A_{\mathbf{P}_i,t}$ with a cone in the Lie algebra $\mathfrak{a}_i$, one can endow it with the simplicial metric d_S defined by the Killing form through the exponential map. Then $(A_{\mathbf{P}_i,t}, d_S)$ is a metric cone over the open simplex $A^+_{\mathbf{P}_i}(\infty)$ in the Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$ associated with $\mathbf{P}_i$, when $A^+_{\mathbf{P}_i}(\infty)$ is endowed with a suitable simplicial metric. We can glue these metric cones $(A_{\mathbf{P}_i,t}, d_S)$ to form a local distance function l_S on $\coprod_{i=0}^n A_{\mathbf{P}_i,t}$. If d_{ind} is the distance function on the subspace $\coprod_{i=0}^n A_{\mathbf{P}_i,t}$ induced by $\Gamma\backslash G/K$, then it is not hard to show that the tangent cone $T_\infty(\coprod_{i=0}^n A_{\mathbf{P}_i,t}, d_{\text{ind}})$ at infinity exists and is equal to $(\coprod_{i=0}^n A_{\mathbf{P}_i,t}, l_S)$. The tangent cone $T_\infty(\Gamma\backslash G/K)$ therefore exists and is equal to a metric cone over the Tits complex $\Delta(\Gamma\backslash G/K)$ [13]. The resulting fact that the Gromov-Hausdorff distance between $\Gamma\backslash G/K$ and $(\coprod_{i=0}^n A_{\mathbf{P}_i,t}, l_S)$ is finite allows us to build a map $\pi : \Gamma\backslash G/K \to (\coprod_{i=0}^n A_{\mathbf{P}_i,t}, l_S)$ whose properties are captured in the following description.

Picture from Reduction Theory: Let $M = \Gamma\backslash G/K$. There is a compact polyhedron Q and a Lipschitz map $\pi : M \to cQ$, where cQ is the open cone on Q so that (1) every point inverse deform retracts to an arithmetic manifold, (2) π respects the radial direction, and (3) all point inverses have uniformly bounded size.

Again, the polyhedron Q is the geometric realization of the category of proper $\mathbb{Q}$-parabolic subgroups of G, modulo the action of Γ. The inverse image $\pi^{-1}(*)$ of the barycenter of a simplex is the arithmetic symmetric space associated to that parabolic. Concretely, for $\mathrm{SL}_n(\mathbb{Z}) \subset \mathrm{SL}_n(\mathbb{R})$, the space Q is an $n-2$ simplex, the parabolics correspond to flags, and the associated arithmetic groups have a unipotent normal subgroup with quotient equal to a product of $\mathrm{SL}_{m_i}(\mathbb{Z})$, where the m_i are sizes of the blocks occurring in the flag. As one goes to infinity, the unipotent directions shrink in diameter and are responsible for the finite volume property of the lattice quotient, while the other parabolic directions remain of bounded size. Alternatively, for any choice of basepoint in the homogeneous space, there are constants C and D that satisfy the following condition: if x is a given point and Q_x is the largest parabolic subgroup associated with a simplex whose cone contains x within its C-neighborhood, then the orbit of x under Q_x has diameter less than D. Note the empty simplex means that there is a compact core which is stabilized by the whole group. In addition to the proof in [13], this picture can be ascertained from [4], [18]; the fact that $\Gamma\backslash G/K$ has finite Gromov-Hausdorff distance from cQ is first asserted in [8].

As a guide the reader should consider the picture suggested by a product of hyperbolic manifolds. In the compact case, each hyperbolic manifold contributes to cQ a point. In the case of cusps, it contributes the open cone on a finite set of points. Thus Q is a join of some number of finite sets. Using this model, we find that the inverse image of any point in the interior of any simplex is exactly a product of closed hyperbolic manifolds, cores of hyperbolic manifolds, and flat manifolds.

To build an appropriate hypersurface in $\Gamma\backslash G/K$, we require a key estimate of Eskin [7] about the "coarse isotropy" of our space (see also [3] and [13]). Some details are provided as follows. Let $P = NAM$ be a minimal parabolic $\mathbb{Q}$-subgroup of G and write $G = NAMK$. Consider the chamber decomposition of $\mathfrak{a}$, the Lie algebra of A. The corresponding Weyl group W acts on these chambers via the hyperplanes. If $G = \coprod_{w\in W} BwB$ is the Bruhat decomposition of G, let $\gamma \in BwB$ for some $w \in W$. If $g = nak$, we write $\gamma g = n'a'k'$. Denote by $\sum^+$ the set of positive roots of $\mathfrak{a}^*$ and $\sum^-$ the set of negative roots. Let $\mathcal{R} = \sum^- \cap w\sum^+$ be the set of roots that are positive but are negated under the action of w. For some positive reals constants c_α, Abels and Margulis [1] provide the following equation:

$$(*) \qquad a' = wa - \sum_{\alpha\in\mathcal{R}} c_\alpha \alpha(a) + o(1),$$

where a and a' are viewed as elements of the Lie algebra $\mathfrak{a}$. The implications of this equation are as follows. Consider an element a in the positive Weyl chamber $\mathcal{C}(a)$. Then the intersection $\Gamma(a) \cap \mathcal{C}(a)$ of the orbit $\Gamma(a)$ of a under Γ and the Weyl chamber $\mathcal{C}(a)$ containing a has a bounded diameter, uniformly in a. In other words, if $\gamma(a)$ stays in the same positive Weyl chamber, then w is the identity and $\mathcal{R}$ is empty. Hence $a' = a + o(1)$, implying that a' can be found at a uniformly bounded distance from a itself. In this event, the action of γ corresponds to a translation of a to (possibly) the compact fiber direction of $\Gamma\backslash G/K$. It is also a general fact that the image of a vertex of any subsector under the action of $\gamma \in \Gamma$ is the vertex of an analogous subsector. With this machinery, we are able to prove the following.

Theorem 4: Let G be a semisimple irreducible Lie group, K its maximal compact subgroup and Γ an arithmetic lattice with $\operatorname{rank}_{\mathbb{Q}}\Gamma \geq 2$. If Γ^* is any torsion-free subgroup of Γ of finite index, then the manifold $X^* = \Gamma^* \backslash G/K$ lacks a uniform positive scalar curvature metric that is coarsely equivalent to the natural one inherited from G.

Proof: Let $G = PK$, where $P = NMA$ is a minimal parabolic $\mathbb{Q}$-subgroup of G. The precise reduction theory provides a compact polyhedron Q and a Lipschitz map $\pi : X^* \to cQ$ from X^* to an open cone cQ on Q. Consider one maximal simplex in Q corresponding to some Weyl chamber $\mathcal{C}^+$, and construct a hypersurface in $\mathcal{C}^+$ as in Figure 1 (this hypersurface is coarsely a cone on a sphere $\mathcal{S}$, where $\mathcal{S}$ lies in the interior of the simplicial face at infinity). This subset H can be oriented so that the distance from H to any hyperplane $\alpha = 0$ will exceed the quantity $\sup_{a\in\mathcal{C}^+} \operatorname{diam}(\Gamma^*(a) \cap \mathcal{C}^+)$, which is finite by $(*)$. Let $W = \pi^{-1}(H)$ be the corresponding hypersurface in X^*; this W induces a coarsely excisive decomposition (Y, Z) of X^*. The fundamental group $\pi_1(W) = NM \cap \Gamma^*$ injects into $\pi_1(X^*) = \Gamma^*$, so the hypothesis of Lemma 3 is satisfied. In the most general case, the space W is coarsely equivalent to a bundle over Euclidean space whose fiber consists of two components: a nilmanifold N^* and (possibly) a compact homogeneous manifold M^*. If M^* is trivial, the argument follows exactly as in Theorem 3. In the presence of a compact homogeneous manifold, we may pass to the coarse index of the Dirac operator to $\mathbb{R} \times M^*$ and use the usual Rosenberg obstruction on M^* as in Theorem 1 to obtain our desired result. The proof that the Gromov-Lawson conjecture holds for compact, locally symmetric manifolds is found in [**?**]. $\square$

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