Fractional Weak Discrepancy and Split Semiorders

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Dedicated to Martin C. Golumbic
on the occasion of his 60th birthday

ABSTRACT

The *fractional weak discrepancy* $wd_F(P)$ of a poset $P = (V, \prec)$ was introduced in [5] as the minimum nonnegative k for which there exists a function $f : V \to \mathbf{R}$ satisfying (i) if $a \prec b$ then $f(a)+1 \leq f(b)$ and (ii) if $a \parallel b$ then $|f(a) - f(b)| \leq k$. In this paper we generalize results in [6, 7] on the range of wd_F for semiorders to the larger class of split semiorders. In particular, we prove that for such posets the range is the set of rationals that can be represented as r/s for which $0 \leq s - 1 \leq r < 2s$.

1 Introduction

In this paper we will consider irreflexive posets $P = (V, \prec)$, and write $x \parallel y$ when elements x and y in V are incomparable. Of particular importance to us will be the posets $\mathbf{r} + \mathbf{s}$ consisting of two disjoint chains, one with r elements

Class of Posets	**Forbidden Subposets**
linear order	no $\mathbf{1}+\mathbf{1}$
weak order	no $\mathbf{2}+\mathbf{1}$
semiorder	no $\mathbf{3}+\mathbf{1}$, no $\mathbf{2}+\mathbf{2}$
interval order	no $\mathbf{2}+\mathbf{2}$

Table 1: Classes of posets characterized in terms of forbidden subposets.

and one with s elements, where $x \parallel y$ whenever x and y are in different chains. For example, the order $\mathbf{3}+\mathbf{1}$ is shown in Figure 1 on Page 4.

We focus on the fractional weak discrepancy of split semiorders and begin with some background on this and related classes of orders. For additional background and context we refer the reader to [1] and [2].

1.1 Split semiorders and related classes

The four classes of posets: linear orders, weak orders, semiorders, and interval orders, are important both because they arise in applications and also because they have elegant characterizations. Each of these classes can be characterized in terms of forbidden subposets of the form $\mathbf{r}+\mathbf{s}$ as detailed in Table 1. Note that this implies the following inclusions:

$$\{\text{linear orders}\} \subseteq \{\text{weak orders}\} \subseteq \{\text{semiorders}\} \subseteq \{\text{interval orders}\}.$$

These classes also have alternative definitions in terms of interval representations. Such representations are useful in constructions as well is in proofs by contradiction. A poset $P=(V,\prec)$ is an *interval order* if each element $v \in V$ can be assigned an interval $I(v)=[L(v),R(v)]$ in the real line so that $x \prec y$ precisely when $I(x)$ is completely to the left of $I(y)$, that is $R(x) < L(y)$. A *semiorder* (*unit interval order*) is an interval order with a representation in which each interval has the same length. By appropriate scaling, we may assume each interval has length 1.

Linear orders and weak orders can also be defined in this way where each element is assigned a real number (i.e., a degenerate interval). A poset $P = (V,\prec)$ is a *linear order* if each $v \in V$ can be assigned a distinct real number $f(v)$ so that $x \prec y$ if and only if $f(x) < f(y)$. A *weak order* is defined similarly except that the values $f(v)$ need not be distinct, so incomparabilities may occur. These representational definitions are illustrated in Table 2.

Observe that for the first three classes in Table 1, the forbidden subposets are those $\mathbf{r}+\mathbf{s}$ where $r \geq 1$, $s \geq 1$, and $r+s=M$ for $M=2,3,4$, respectively. Such orders are called $(M,2)$-*free* in [10]. More generally, an order is (M,t)-*free* if it contains no poset of the form $\mathbf{r_1}+\mathbf{r_2}+\cdots+\mathbf{r_t}$ where $r_1+r_2+\cdots+r_t=M$.

A next natural class to consider is the class of $(5,2)$-free posets, that is, the posets characterized as having no induced $\mathbf{4}+\mathbf{1}$ and no induced $\mathbf{3}+\mathbf{2}$. This

Class of Posets	v assigned $I_v = [L(v), R(v)]$	$x \prec y$ iff
interval order		$R(x) < L(y)$
semiorder	$R(v) = L(v) + 1$	$R(x) < L(y)$
weak order	$f(v) = L(v) = R(v)$	$f(x) < f(y)$
linear order	$f(v) = L(v) = R(v),\ f(x) \neq f(y)$ for $x \neq y$	$f(x) < f(y)$

Table 2: Classes of posets characterized in terms of representations.

class is called the *subsemiorders* in [1]. Unfortunately, the class of subsemiorders has no known characterization in terms of representations, thus we instead consider a subclass called split semiorders.

Definition 1 A poset $P = (V, \prec)$ is a *split semiorder* if each $v \in V$ can be assigned an interval $I(v) = [L(v), R(v)]$ of unit length u (with $R(v) = L(v) + u$) and a point $C(v) \in I(v)$ so that $x \prec y$ if and only $C(x) < L(y)$ and $R(x) < C(y)$. The point $C(v)$ is called the *point core* or *splitting point* of the interval $I(v)$ and the representation is called a *unit point-core representation.*

Given a unit point-core representation of a split semiorder, a comparability occurs between elements x and y precisely when neither interval $I(x), I(y)$ contains the other interval's splitting point. In the literature on tolerance graphs, split semiorders are also referred to as unit point-core bitolerance orders [4].

Any representation of a poset by real intervals is said to be *unit* if all the intervals in the representation have the same length and *proper* if no interval properly contains another. Sometimes a proper representation is more convenient to construct than a unit representation and thus the following remark can be helpful. Its proof follows from Theorem 10.3 of [4].

Remark 2 A poset P is a split semiorder if and only if it satisfies Definition 1 with a proper representation by intervals $I(v)$ and splitting points $C(v)$ rather than a unit representation.

Every semiorder P has a unit point-core representation obtained by supplementing any unit interval representation P with a point-core assignment C such that $C(v) = L(v)$ for all $v \in V$. Thus, *every semiorder is a split semiorder.* However the containment is proper since $\mathbf{3} + \mathbf{1}$ is a split semiorder that is not a semiorder (see Figure 1). The posets $\mathbf{4} + \mathbf{1}$ and $\mathbf{3} + \mathbf{2}$ are not split semiorders. The details of these proofs appear in [2] and also in Chapter 10 of [4]. Thus split semiorders are indeed $(5, 2)$-free.

We will need the following basic facts about split semiorders in the proof of Proposition 14, in Section 2.3.

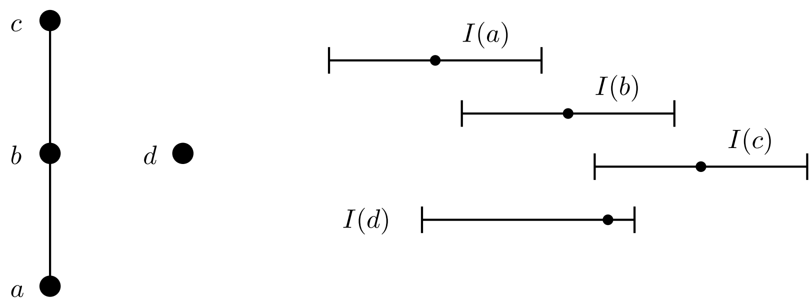

Figure 1: The order $\mathbf{3}+\mathbf{1}$ and a representation of it as a split semiorder.

Lemma 3 *Let P be a split semiorder with a unit point-core representation and let $v \parallel w$ in P.*

(a) $L(v) \leq R(w)$ and $L(w) \leq R(v)$.

(b) If $t \prec u \prec v \parallel w$ in P, then $R(t) < R(w)$ and $C(t) < C(w)$.

(c) If $t \prec u \prec v \parallel w \prec x$, then $t \prec x$.

Proof. Since $w \not\prec v$, by Definition 1 either (i) $C(v) \leq R(w)$ or (ii) $L(v) \leq C(w)$.

(a) In case (i) we have $L(v) \leq C(v) \leq R(w)$. In case (ii) we have $L(v) \leq C(w) \leq R(w)$. So $L(v) \leq R(w)$ is true in both cases and $L(w) \leq R(v)$ follows by symmetry.

(b) In case (i) we have $R(t) < C(v) \leq R(w)$. Similarly, $R(u) < C(v) \leq R(w)$ and since this is a unit representation, $L(u) < L(w)$. Thus

$$C(t) < L(u) < L(w) \leq C(w).$$

In case (ii), again by Definition 1, we have

$$C(t) \leq R(t) < C(u) < L(v) \leq C(w) \leq R(w).$$

So in both cases $R(t) < R(w)$ and $C(t) < C(w)$.

(c) Now suppose we also have $w \prec x$. If $x \prec t$ then $w \prec v$, which contradicts $v \parallel w$. If $t \parallel x$ then it is straightforward to check that the chains $t \prec u \prec v$ and $w \prec x$ form a $\mathbf{3}+\mathbf{2}$ in P. Since P is a split semiorder it is $(5,2)$-free, so this is a contradiction. Thus $t \prec x$. $\square$

1.2 Fractional Weak Discrepancy

For a weak order $P = (V, \prec)$, we can think of the function $f : V \to \mathbf{R}$ as ranking the elements in a way that respects the ordering $\prec$ and gives incomparable elements equal rank. For posets in general, we can try to minimize the discrepancy

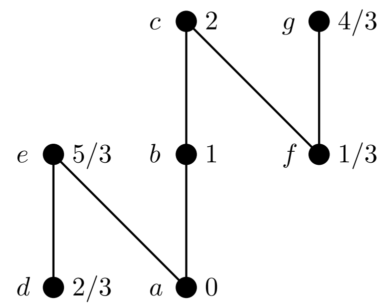

Figure 2: The poset Z with a labeling satisfying the conditions of Definition 4 with $k = 4/3$.

in ranks between incomparable elements. This notion is made more formal in the following definition.

Definition 4 The *fractional weak discrepancy* $wd_F(P)$ of a poset $P = (V, \prec)$ is the minimum nonnegative real number k for which there exists a function $f : V \to \mathbf{R}$ satisfying

(i) if $a \prec b$ then $f(a) + 1 \leq f(b)$ ("up" constraints)

(ii) if $a \parallel b$ then $|f(a) - f(b)| \leq k$. ("side" constraints)

Such a function is called an *optimal labeling* of P.

To illustrate this definition, Figure 2 shows a poset Z with a labeling function that satisfies conditions (i) and (ii) for $k = 4/3$, thus $wd_F(Z) \leq 4/3$. We will show later that this is indeed an optimal labeling and thus $wd_F(Z) = 4/3$.

Fractional weak discrepancy was first defined in [5] and studied further in [6, 7, 8]. The integer version of the problem (where each function value $f(v)$ must be an integer) was introduced in [10] as the *weakness* of a poset, and studied further as *weak discrepancy* in [3, 9]. The poset $\mathbf{3} + \mathbf{1}$ shown in Figure 1 has weak discrepancy and fractional weak discrepancy equal to 1 with the following optimal labeling: $f(a) = 0$, $f(b) = 1$, $f(c) = 2$, $f(d) = 1$. Furthermore, any poset P containing an induced $\mathbf{3} + \mathbf{1}$ will have $wd_F(P) \geq 1$.

The existence of a labeling of a poset P satisfying conditions (i) and (ii) of Definition 4 for a particular k shows that $wd_F(P) \leq k$. We seek a certificate to demonstrate that a labeling is optimal in the form of a substructure that ensures $wd_F(P) \geq k$.

Forcing cycles, which we define now, provide our main tool for proving results about fractional weak discrepancy. Theorem 6 shows that $wd_F(P)$ can be calculated from an appropriate forcing cycle.

Definition 5 A *forcing cycle* C of poset $P = (V, \prec)$ is a sequence $C : x_0, x_1, \ldots, x_m = x_0$ of $m \geq 2$ elements of V for which $x_i \prec x_{i+1}$ (an *up step*) or $x_i \parallel x_{i+1}$ (a *side step*) for each $i : 0 \leq i < m$. If C is a forcing cycle, we write $\mathrm{up}(C) = |\{i : x_i \prec x_{i+1}\}|$ and $\mathrm{side}(C) = |\{i : x_i \parallel x_{i+1}\}|$.

Note that all forcing cycles C have $\mathrm{up}(C) \geq 0$ and $\mathrm{side}(C) \geq 2$.

Theorem 6 [5] Let $P = (V, \prec)$ be a poset with at least one incomparable pair. Then $wd_F(P) = \max_C \frac{\mathrm{up}(C)}{\mathrm{side}(C)}$, where the maximum is taken over all forcing cycles C in P.

Definition 7 If $wd_F(P) = \frac{\mathrm{up}(C)}{\mathrm{side}(C)}$, we call C an *optimal forcing cycle* of P.

For example, it easy to check that the poset $P = \mathbf{3} + \mathbf{1}$ of Figure 1 has three forcing cycles. The cycle $a \prec b \prec c \parallel d \parallel a$ gives the maximum ratio $\frac{\mathrm{up}(C)}{\mathrm{side}(C)}$ and is thus optimal, with $wd_F(P) = 2/2 = 1$.

In general, the up-to-side ratio for any forcing cycle gives a lower bound for the fractional weak discrepancy. For example, the poset Z of Figure 2 has many forcing cycles. The cycle $a \prec b \prec c \parallel d \prec e \parallel f \prec g \parallel a$ shows that $wd_F(Z) \geq 4/3$. The labeling in the figure shows that $wd_F(Z) \leq 4/3$. Thus this forcing cycle is optimal and $wd_F(Z) = 4/3$.

Once a starting point is specified, a forcing cycle can be described as p alternating sequences U_j of u_j consecutive up steps and S_j of s_j consecutive side steps for $j = 1, 2, \ldots, p$. Thus $\mathrm{up}(C) = \sum_1^p u_j$ and $\mathrm{side}(C) = \sum_1^p s_j$. For example, the optimal forcing cycle we found for Z has $p = 3$ with $u_1 = 2$, $s_1 = 1$, $u_2 = 1$, $s_2 = 1$, $u_3 = 1$, and $s_3 = 1$. This notation will be useful in Section 2.

Theorem 6 implies that the fractional weak discrepancy of any poset will be a rational number, but which rational numbers are actually achieved? In this paper we fully answer this question for split semiorders.

2 An upper bound for wd_F of a split semiorder

In this section we give an upper bound for the fractional weak discrepancy of a split semiorder. In [6] we proved that $wd_F(P) < 1$ if and only if P is a semiorder. In Corollary 9, we prove a similar result for split semiorders.

Theorem 8 *Let P be a split semiorder and C be a forcing cycle in P. Then* $\mathrm{up}(C) \leq 2(\mathrm{side}(C) - 1)$.

Corollary 9 *For any split semiorder P, we have $wd_F(P) < 2$.*

Corollary 9 will follow from Theorem 8, since by applying Theorem 6 to an *optimal* forcing cycle C we find $wd_F(P) = \frac{\mathrm{up}(C)}{\mathrm{side}(C)} \leq 2\left(1 - \frac{1}{\mathrm{side}(C)}\right) < 2$. We will see by results in Section 3 that the upper bounds in Theorem 8 and Corollary 9 are the best possible ones for split semiorders.

The rest of this section is devoted to proving Theorem 8. We will assume an instance where it is false for some C. We then apply an algorithm that moves along the cycle through successive sequences of up steps and of side steps and builds a stack K of elements of C. Finally we derive a contradiction from K, thus completing the proof of the theorem.

Throughout the remainder of Section 2 we will make the following *background assumptions* (BA) for the algorithm:

$$\begin{aligned}&P \text{ is a split semiorder with a fixed unit point-core representation,}\\ &C \text{ is a forcing cycle in } P,\\ &r = \mathrm{up}(C),\ s = \mathrm{side}(C), \text{ and } r > 2(s-1).\end{aligned} \tag{BA}$$

2.1 The algorithm

The algorithm consists of three stages; in expressing them we make several assertions, which we prove in Sections 2.2 and 2.3. The stages are:

1. Preprocessing: Let C consist of p alternating sequences of u_j consecutive up steps and s_j consecutive side steps, $j = 1, 2, \dots, p$. If necessary, relabel C to start the cycle at the beginning of a sequence of up steps for which the partial sums of $\Sigma_{j=1}^{p}(u_j - \lambda_j)$ are nonnegative, where

$$\lambda_j = \begin{cases} 2s_j - 1, & \text{if } j = p \\ 2s_j, & \text{otherwise.} \end{cases} \tag{1}$$

2. Initialization: (*phase* 0) Initialize the stack K with the first element of C.

3. Iteration: For each $j = 1, 2, \dots, p$,

   jth up-step phase: Add (push) the next u_j elements of C, corresponding to the next sequence U_j of up steps, to the top of K.

   jth side-step phase: Remove (pop) the top λ_j elements from K.

We iterate these phases until we return to the beginning of C. We first prove (Proposition 12) that there exists a starting point for C as described in the preprocessing step. We use this to prove (Proposition 13) that the stack never empties during the iteration. We then use the unit point-core representation of P to prove (Proposition 14) that after each step of the algorithm, the order of elements on the stack K respects the partial order of P. Finally, we use this structural property of K to prove Theorem 8 by showing that C is not a forcing cycle, which contradicts our background assumptions (BA).

2.2 Preprocessing to obtain a good starting point

We must show that there exists a labeling of the forcing cycle C for which the partial sums of $\sum_{j=1}^{p}(u_j - \lambda_j)$, as defined in Section 2.1, are all nonnegative.

Lemma 10 *Under the background assumptions in* (BA), $\sum_{j=1}^{p}(u_j - \lambda_j) \geq 0$.

Proof. Since $r > 2(s-1)$ and $s = \sum_{j=1}^{p} s_j$, equation (1) implies

$$\begin{aligned}\sum_{j=1}^{p}(u_j - \lambda_j) &= \sum_{j=1}^{p} u_j - \sum_{j=1}^{p} \lambda_j = r - \sum_{j=1}^{p} 2s_j + 1 \\ &> 2(s-1) - 2s + 1 \\ &= -1.\end{aligned}$$

Since both sides are integers the result follows. □

We will also need the fact that whenever the sum of a finite number of real numbers is nonnegative, there is a cyclic permutation of the terms that makes all the partial sums nonnegative. This fact is a variant of a result proved in [8].

Lemma 11 [8] *Let $t_1, t_2, \ldots, t_p$ be a finite sequence of real numbers with $\sum_{j=1}^{p} t_j \geq 0$. There exists an index q with $1 \leq q \leq p$ so that the partial sums of the sequence $t_{q+1}, t_{q+2}, \ldots, t_p, t_1, t_2, \ldots, t_q$ are all nonnegative.*

We may choose to start the cycle at an element x_0 that is the beginning of a sequence of up steps, i.e., if C contains m elements then $x_{m-1} \parallel x_m = x_0 \prec x_1$. We call x_0 an *upward starting point* for C.

The existence of the required labeling now follows by applying Lemma 11 to the sequence $\{u_j - \lambda_j\}$ and letting the new starting point of C be $x_{u_1+s_1+\cdots+u_q+s_q}$. This proves the following result and completes the preprocessing step of the algorithm.

Proposition 12 *Under the background assumptions in* (BA)*, there is an upward starting point for C for which the partial sums of $\sum_{j=1}^{p}(u_j - \lambda_j)$ are all nonnegative.*

2.3 Initialization and iteration

We initialize the stack K with the upward starting point x_0 and then for $j = 1, 2, \ldots, p$ we push the next sequence of u_j elements and then pop λ_j elements.

We will use the following notation to help describe the evolution of the stack K during the algorithm. This is summarized in Table 3 along with other notation from this section. Let β_j be the first element added to the stack during the jth up-step phase and let α_j be the top element of the stack after the jth side-step phase. Denote the elements on the stack after the jth up-step phase, from the top of the stack down, by $b_1, b_2, \ldots$. Then $b_{u_j} = \beta_j$ and the top u_j elements of K correspond to the j^{th} sequence of up steps in C, namely $U_j : \beta_j = b_{u_j} \prec \cdots \prec b_2 \prec b_1$.

In the forcing cycle C, U_j is followed by s_j elements corresponding to the next sequence of side steps, $S_j : d_1 \parallel d_2 \parallel \cdots \parallel d_{s_j}$. We remark that b_1 and d_1 depend on j, but we have suppressed this dependence in the notation. At the two ends of S_j we have

$$b_1 \parallel d_1 \quad \text{for} \quad 1 \leq j \leq p, \tag{2}$$

$$d_{s_j} \prec \beta_{j+1} \quad \text{for} \quad 1 \leq j \leq p-1. \tag{3}$$

Proposition 13 *Under the background assumptions in* (BA)*, the stack K never empties during the algorithm.*

Proof. The number of elements on the stack after the jth up-step phase of the algorithm is $1 + \sum_{l=1}^{j-1}(u_l - \lambda_l) + u_j$. The number after the succeeding jth side-step phase is $1+\sum_{l=1}^{j}(u_l-\lambda_l)$. By Proposition 12, there are always at least two elements on the stack after the jth up-step phase and at least one after the jth side-step phase. Thus the stack never empties during the algorithm. □

Proposition 14 *Under the background assumptions in* (BA)*, after each phase of the algorithm the elements of the stack K form a chain that respects the partial order in P.*

Proof. Since in the initialization phase only one element is placed on the stack and during side-step phases the algorithm only pops elements, it is enough to prove the result just for the jth up-step phase. We will do this by induction on j.

For the case $j = 1$, the result is true since x_0 is an upward starting point for C. Now suppose the result is true for $1, 2, \ldots, j$, where $1 \leq j \leq p-1$, and prove it is true for $j+1$. We consider the $(j+1)$st up-step phase. It suffices to prove that

$$\alpha_j \prec \beta_{j+1}. \tag{4}$$

By the induction assumption for j, after the jth up-step phase the stack K is a chain

$$\cdots \prec b_3 \prec b_2 \prec b_1. \tag{5}$$

Since we then popped $\lambda_j = 2s_j$ elements in the jth side-step phase and the stack never empties, the chain (5) contains at least three elements. We popped $b_1, b_2, \ldots, b_{2s_j}$, so the next element on the stack is

$$\alpha_j = b_{2s_j+1}. \tag{6}$$

We can think of this process as removing s_j pairs of elements, one pair at a time. If we remove only $s_j = 1$ pair then $\alpha_j = b_3 \prec b_2 \prec b_1 \parallel d_1 \prec \beta_{j+1}$, by (2) and (3). By Lemma 3(c), relation (4) follows. This completes the induction step when $s_j = 1$.

Now suppose $s_j \geq 2$. When we have removed i pairs, $1 \leq i \leq s_j$, we let $e_i = b_{2i+1}$ denote the element at the top of the stack at that point. In order to prove (4), we will compare the endpoints and splitting points of the intervals $I(e_i)$ and $I(d_i)$ as we pop pairs from the stack. In particular, we will prove by a second induction on i that

$$R(e_i) < R(d_i) \quad \text{and} \quad C(e_i) < C(d_i) \quad \text{for} \quad 1 \leq i \leq s_j. \tag{7}$$

j^{th} sequence of up steps in C	$U_j : b_{u_j} \prec \cdots \prec b_2 \prec b_1$
j^{th} sequence of side steps in C	$S_j : d_1 \parallel d_2 \parallel \cdots \parallel d_{s_j}$
After sequence U_j is processed the top of stack K is	$\beta_j = b_{u_j} \prec \cdots \prec b_2 \prec b_1$
After sequence S_j is processed the top of stack K is	α_j
Definition of e_i	$e_i = b_{2i+1}$
From (6) and the definition of e_i	$\alpha_j = b_{2s_j+1} = e_{s_j}$ for $1 \leq j \leq p-1$

Table 3: Summary of notation used in Section 2.

When $i = 1$, $e_1 = b_3 \prec b_2 \prec b_1 \parallel d_1$. So Lemma 3(b) proves that (7) is true in this base case.

Now suppose that $i \geq 2$ and that (7) is true for $1, 2, \ldots, i-1$. We will prove it is true for i. Since $d_{i-1} \parallel d_i$, we know $d_i \not\prec d_{i-1}$. So by Definition 1 either (i) $C(d_{i-1}) \leq R(d_i)$ or (ii) $L(d_{i-1}) \leq C(d_i)$.

In case (i), the induction assumption (7) for $i-1$ together with $e_i = b_{2i+1} \prec b_{2i} \prec b_{2i-1} = e_{i-1}$ imply that

$$R(e_i) < C(b_{2i}) \leq R(b_{2i}) < C(e_{i-1}) < C(d_{i-1}) \leq R(d_i).$$

In particular $R(b_{2i}) < R(d_i)$, and since the representation is unit we also have $L(b_{2i}) < L(d_i)$. Thus,

$$C(e_i) < L(b_{2i}) < L(d_i) \leq C(d_i).$$

So for case (i), this proves i satisfies (7).

In case (ii), again note that (7) for $i-1$ implies $R(e_{i-1}) < R(d_{i-1})$ and therefore $L(e_{i-1}) < L(d_{i-1})$. Thus using Definition 1,

$$C(e_i) < L(e_{i-1}) < L(d_{i-1}) \leq C(d_i).$$

Also,

$$R(e_i) < C(b_{2i}) < L(e_{i-1}) \leq C(d_i) \leq R(d_i).$$

This proves i satisfies (7) for case (ii) and completes the induction on i, the number of pairs popped in the jth side-step phase.

We now return to the induction on j, where it remains to prove (4), i.e., $\alpha_j \prec \beta_{j+1}$. Recall from (6) that $e_{s_j} = \alpha_j$ and from (3) that $d_{s_j} \prec \beta_{j+1}$. Thus using (7), we have $R(e_{s_j}) < R(d_{s_j}) < C(\beta_{j+1})$ and $C(e_{s_j}) < C(d_{s_j}) < L(\beta_{j+1})$. We conclude that $\alpha_j \prec \beta_{j+1}$ as required. □

Note that in the preceding argument we proved (7) for $1 \leq i \leq s_j$ when $1 \leq j \leq p-1$. In fact the argument is equally valid when $j = p$ provided $1 \leq i \leq s_p - 1$, since then we pop $s_p - 1$ pairs of elements from the stack and then one final element. We will make use of this fact in the proof of Theorem 8.

2.4 Proof of Theorem 8

Now that we have verified the algorithm has the desired properties, we go on to prove Theorem 8 by contradiction. Let $x_0, x_1, \ldots, x_m$ be the elements of the forcing cycle C. We have assumed $r > 2(s-1)$ in the algorithm, where $r = \text{up}(C)$ and $s = \text{side}(C)$. We now consider the possible forms of the stack K after the final (pth) side-step phase. By the initialization phase and Proposition 13, the bottom element of K is x_0. We consider the cases $s_p = 1$ and $s_p \geq 2$ separately.

Suppose $s_p = 1$, that is, the last sequence S_p of side steps consists of exactly one side step. Since x_0 is an upward starting point for C we then have $x_{m-2} \prec x_{m-1} \parallel x_m = x_0$. After the pth up-step phase, the element at the top of K is x_{m-1} and the element on the bottom is x_0. By Proposition 14, it follows that $x_0 \prec x_{m-1}$, a contradiction.

Now suppose $s_p \geq 2$, that is, S_p contains at least two side steps. In the pth side-step phase we remove $\lambda_p = 2s_p - 1$ elements from the top of K without emptying it, $s_p - 1$ pairs of elements $b_1, \ldots, b_{2s_p-2}$ and then the single element $b_{2s_p-1} = e_{s_p-1}$ that is still at the top. So after the pth up-step phase that precedes it, the stack K consists of at least the top $2s_p$ elements listed in (5).

In addition, x_0 is on the bottom of the stack (and may equal b_{2s_p}). By (7) applied in the case $j = p$ and $i = s_p - 1$, it must be the case that

$$R(e_{s_p-1}) < R(d_{s_p-1}) \quad \text{and} \quad C(e_{s_p-1}) < C(d_{s_p-1}).$$

Because the representation is unit, the first inequality implies $L(e_{s_p-1}) < L(d_{s_p-1})$.

Since $b_{2s_p} \prec b_{2s_p-1} = e_{s_p-1}$, we have $C(b_{2s_p}) < L(e_{s_p-1}) < L(d_{s_p-1})$. Similarly, $R(b_{2s_p}) < C(e_{s_p-1}) < C(d_{s_p-1})$. Thus, $b_{2s_p} \prec d_{s_p-1} \parallel d_{s_p} = x_0$. This contradicts the fact that either $x_0 = b_{2s_p}$ or $x_0 \prec b_{2s_p}$.

Since all possible forms of K after the last (pth) side-step phase lead to a contradiction, it follows that $r \leq 2(s-1)$. This completes the proof of Theorem 8. $\square$

3 The range of wd_F for split semiorders

In the preceding section, Theorem 8 gave an upper bound for the range of the wd_F function for split semiorders. Our goal in this section is to determine the values that make up the range. In particular, we will prove (Theorem 16) that for each rational number r/s for which r is in an interval determined by s, there exists a split semiorder whose fractional weak discrepancy equals r/s and an optimal forcing cycle C with $\text{up}(C) = r, \text{side}(C) = s$. The proof is constructive.

It is possible for $wd_F(P)$ to be equal to some fraction r/s in lowest terms but for there to be no optimal forcing cycle C with $\text{up}(C) = r, \text{side}(C) = s$. In this case each optimal C will have $\text{up}(C) = lr, \text{side}(C) = ls$ for some integer $l > 1$. We will give an example of this after the proof of Corollary 18.

In the proofs of Theorem 16 and Corollary 18 we will refer to the following, which combines results proved in [6].

Theorem 15 [6] *A poset P is a semiorder if and only if $wd_F(P) < 1$. If P is a semiorder then $wd_F(P) = \frac{r}{r+1}$ for some integer $r \geq 0$. Moreover, for each integer $r \geq 0$ there exists a semiorder P with $wd_F(P) = \frac{r}{r+1}$ and an optimal forcing cycle C with* $\mathrm{up}(C) = r, \mathrm{side}(C) = r+1$.

Theorem 16 *Let r, s be integers for which $s \geq 2$ and $s-1 \leq r \leq 2(s-1)$. There exists a split semiorder P with $wd_F(P) = r/s$ and an optimal forcing cycle C having* $\mathrm{up}(C) = r,\ \mathrm{side}(C) = s$.

Proof. Let $s \geq 2$. For $r = s-1$, Theorem 15 implies that there exists a semiorder P with $wd_F(P) = r/s$ and the desired forcing cycle. Since P is also a split semiorder we have proved the theorem for the case $r = s-1$.

Now assume that $s \leq r \leq 2(s-1)$. We begin by constructing a unit point-core representation for a split semiorder $P = (V, \prec)$ possessing a forcing cycle C with $\mathrm{up}(C) = r,\ \mathrm{side}(C) = s$. After that, we will show C is optimal.

Constructing a split semiorder P and forcing cycle C. Begin by setting $V = \{x_0, x_1, \ldots, x_r, y_1, y_2, \ldots, y_{s-1}\}$. Define

$$q = \frac{r}{2s-r-1}. \tag{8}$$

Notice that $2s-r-1 = 2(s-1)-r+1 \geq 1$, since we have assumed $2(s-1) \geq r$. Also $2s-r-1 \leq 2r-r-1 = r-1$ since $s \leq r$. Thus,

$$1 < \frac{r}{r-1} \leq \frac{r}{2s-r-1} = q \leq r.$$

For $0 \leq i \leq r$, define $I(x_i) = [L(x_i), R(x_i)]$ with splitting point $C(x_i)$ by

$$L(x_i) = i(q+1), \qquad C(x_i) = i(q+1)+q, \qquad R(x_i) = i(q+1)+2q. \tag{9}$$

Similarly, for $1 \leq j \leq s-1$ define $I(y_j)$ by

$$L(y_j) = 2jq, \qquad C(y_j) = 2jq, \qquad R(y_j) = 2(j+1)q. \tag{10}$$

This collection of intervals $I(x_i), I(y_j)$ and splitting points $C(x_i), C(y_j)$ gives a representation of a split semiorder $P = (V, \prec)$. Note that all the intervals have length $2q > 2$, and that the splitting point of $I(x_i)$ is at its midpoint while that of $I(y_j)$ is at its left endpoint.

Figure 3 shows a Hasse diagram for P when $r = 6, s = 4$, so $q = 6$, and Figure 4 illustrates the unit point-core representation we have constructed for it. The function g shown in the figure is an auxiliary function that will be used on Page 14 to obtain an optimal labeling of P.

We will now prove that $C : x_0, x_1, \cdots, x_r, y_{s-1}, y_{s-2}, \cdots, y_1, x_0$ is a forcing cycle. By Definition 1, equations (9) and (10) imply directly that $x_i \prec x_{i+1}$

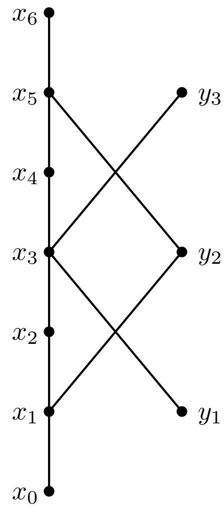

Figure 3: A Hasse diagram for the poset P with $r = 6, s = 4, q = 6$. Figure 4 gives a unit point-core representation for P, showing it is a split semiorder.

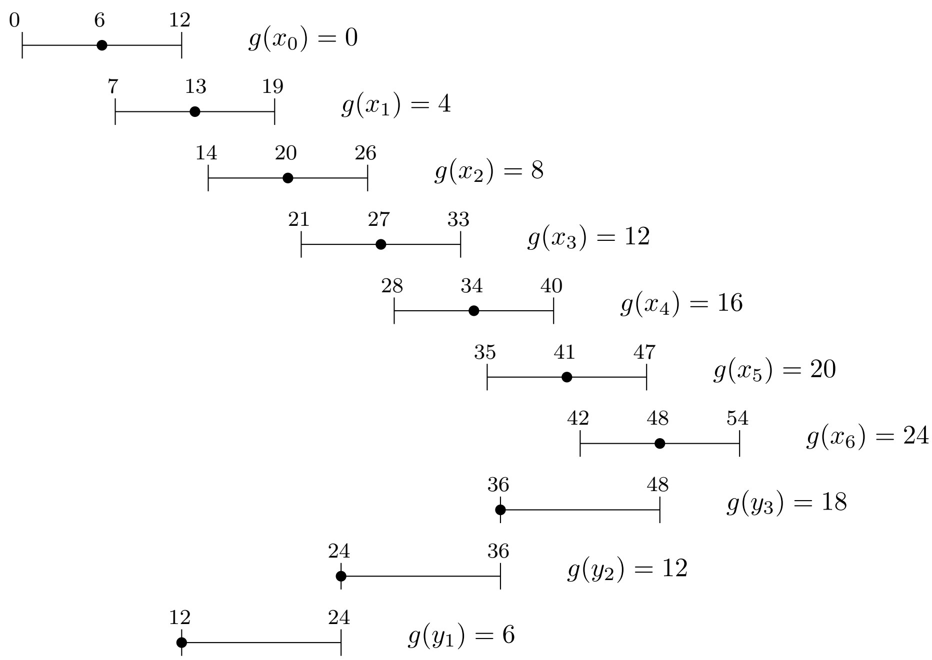

Figure 4: A unit point-core representation for the poset P in Figure 3. Here g is an auxiliary function used to obtain an optimal labeling.

for $0 \le i < r$ and $y_j \parallel y_{j+1}$ for $1 \le j \le s-2$. Furthermore, $y_1 \parallel x_0$ since $R(x_0) = 2q = C(y_1)$. Finally, $x_r \parallel y_{s-1}$ because by (8) we have

$$\begin{aligned} C(x_r) &= r(q+1) + q = (r+1)q + r = \frac{(r+1)r}{2s-r-1} + r \\ &= \frac{2sr}{2s-r-1} = 2sq = R(y_{s-1}). \end{aligned} \tag{11}$$

Thus $x_0 \prec x_1 \prec \cdots \prec x_r \parallel y_{s-1} \parallel y_{s-2} \parallel \cdots \parallel y_1 \parallel x_0$, and C is a forcing cycle with $\mathrm{up}(C) = r$ and $\mathrm{side}(C) = s$. In particular, Theorem 6 implies that $wd_F(P) \ge r/s$.

Before we define a labeling and prove it is optimal, it will be useful to express the endpoints and splitting points of the x- and y-intervals terms of r and s. By (8), we have

$$\begin{aligned} L(x_i) &= i(q+1) = i\left(\frac{r}{2s-r-1} + 1\right) = \frac{2is-i}{2s-r-1} \\ C(x_i) &= i(q+1) + q = \frac{2is-i+r}{2s-r-1} \\ R(x_i) &= i(q+1) + 2q = \frac{2is-i+2r}{2s-r-1} \\ L(y_j) &= C(y_j) = 2jq = \frac{2jr}{2s-r-1} \\ R(y_j) &= 2(j+1)q = \frac{2jr+2r}{2s-r-1}. \end{aligned} \tag{12}$$

An optimal labeling f. We require a labeling of the elements of P that satisfies Definition 4 with $k = r/s$. Let

$$\begin{aligned} g(x_i) &= is, \quad i = 0, 1, \ldots, r \\ g(y_j) &= jr, \quad j = 1, 2, \ldots, s-1. \end{aligned}$$

Then define the labeling $f : V \to \mathbf{Q}$ by $f(u) = g(u)/s$, i.e.,

$$\begin{aligned} f(x_i) &= i \\ f(y_j) &= j\frac{r}{s}. \end{aligned}$$

For example, in the split semiorder P shown in Figures 3 and 4, $C : x_0 \prec x_1 \prec \cdots \prec x_6 \parallel y_3 \parallel y_2 \parallel y_1 \parallel x_0$ is a forcing cycle. Since $\mathrm{up}(C) = 6, \mathrm{side}(C) = 4$, Theorem 6 implies $wd_F(P) \ge 6/4$. The values of $g(u)$ are shown in Figure 4 and $f(u) = g(u)/4$ satisfies Definition 4 with $k = 6/4$. Thus $wd_F(P) \le 6/4$ and by combining the two inequalities we see $wd_F(P) = 6/4 = 3/2$.

We will prove that f satisfies Definition 4 in general, with $k = r/s$. This will show $wd_F(P) \le r/s$ and thus that $wd_F(P) = r/s$. So it suffices to prove

(i) if $a \prec b$ then $g(a) + s \le g(b)$ (“up” constraints)

(ii) if $a \parallel b$ then $|g(a) - g(b)| \leq r$. ("side" constraints)

There are several cases to consider, and we will see that it may be necessary to modify the construction of P and its interval representation in order to complete the proof.

We start with the "side" constraints (ii), as they are easier to prove. Since $x_i \parallel x_j$ if and only if $i = j$, there are only two cases to consider.

Case $y_i \parallel y_j$. It is straightforward to check that when $i \leq j$, then $y_i \parallel y_j$ if and only if $j = i + 1$ and that $|g(y_{i+1}) - g(y_i)| = r$

Case $x_i \parallel y_j$. Let $x_i \parallel y_j$. We will prove that $|g(x_i) - g(y_j)| \leq r$. By Lemma 3(a),

$$2jq = L(y_j) \leq R(x_i) = i(q+1) + 2q. \tag{13}$$

Also, either $L(x_i) \leq C(y_j) = L(y_j)$ or $C(x_i) \leq R(y_j)$. Thus, either

$$i(q+1) \leq 2jq \quad \text{or} \quad i(q+1) + q \leq 2(j+1)q = 2jq + 2q$$

and so in any case

$$i(q+1) \leq 2jq + q.$$

Combining this with (13) we obtain

$$i(q+1) \leq (2j+1)q \leq i(q+1) + 3q = (i+3)q + i.$$

Substituting $q = \frac{r}{2s-r-1}$ from (8) and noting that $i \leq r$, we find

$$\begin{aligned}
i\left(\frac{r}{2s-r-1} + 1\right) \leq (2j+1)\frac{r}{2s-r-1} &\leq (i+3)\frac{r}{2s-r-1} + i \\
ir + 2is - ir - i \leq (2j+1)r &\leq (i+3)r + 2is - ir - i \\
2is - i \leq 2jr + r &\leq 3r + 2is - i \\
-r - i \leq 2jr - 2is &\leq 2r - i \\
-r \leq -\frac{r+i}{2} \leq jr - is &\leq r - \frac{i}{2} \leq r.
\end{aligned}$$

This proves $|g(x_i) - g(y_j)| = |is - jr| \leq r$, as desired.

We now return to the "up" constraints (i), where we want to show that if $a \prec b$ then $g(a) + s \leq g(b)$.

Case $x_i \prec x_j$. Let $x_i \prec x_j$, i.e., $i < j$. Then $g(x_i) + s = (i+1)s \leq js = g(x_j)$.

Case $y_i \prec y_j$. Let $y_i \prec y_j$, i.e., $i \leq j - 2$. Since $s \leq r$,

$$g(y_i) + s = ir + s \leq (j-2)r + r = (j-1)r < jr = g(y_j).$$

The remaining cases; modifying the order P. In the remaining two "up" cases, $x_i \prec y_j$ and $y_j \prec x_i$, constraint (i) may not always be true, and it may therefore be necessary to alter slightly some of the intervals in the representation. This will change the poset $P = (V, \prec)$ by removing some comparabilities between pairs of elements and may also destroy the unit property of the representation. However, we will show that the new representation is proper, so Remark 2 will imply that the resulting poset $P' = (V, \prec')$ is a split semiorder. We will remove comparabilities in a way that will not affect any other pair of elements, so the conclusions we drew in the four cases considered so far will remain valid. This will not change the forcing cycle C. We will see that P' satisfies properties (i) and (ii) of Definition 4 for all pairs of elements, so it will have the properties required by Theorem 16. We now consider the two "up" cases that remain.

Case $x_i \prec y_j$. We must now consider all relations of the form $x_i \prec y_j$. We proceed by sweeping through the intervals $I(x_i)$ from right to left, i.e., with $i = r, r-1, \ldots, 1, 0$. For a given i, suppose $x_i \prec y_j$ for some y_j. Either we will prove that (i) is true or else we will redefine $L(y_j)$ and $C(y_j)$ by moving them to the left in a way that satisfies the constraints. This change will not affect the validity of the constraints for any i previously considered, i.e., for any larger value of i, so we may continue moving from right to left even when we modify the representation.

We first show that $i \leq r-2$, i.e., this case cannot occur in the first two steps at the start of the sweeping process. Since the right endpoints of the x-intervals and the splitting points of the y-intervals are strictly increasing, it suffices to show that $R(x_{r-1}) \geq C(y_{s-1})$ and thus $x_{r-1} \not\prec y_{s-1}$. By (12) we have

$$C(y_{s-1}) = r\left(\frac{2s-2}{2s-r-1}\right),$$

$$R(x_{r-1}) = \frac{(r-1)(2s-1)+2r}{2s-r-1} = \frac{r(2s-2)+(3r-2s+1)}{2s-r-1}.$$

Since we have assumed $2 \leq s \leq r$ we know $3r-2s+1 \geq s+1 \geq 3$. So $i \leq r-2$.

Next we establish that

$$g(x_i) = is < jr = g(y_j). \tag{14}$$

Since $x_i \prec y_j$ we have $R(x_i) < C(y_j) = L(y_j)$, i.e. by (12)

$$R(x_i) = \frac{2is - i + 2r}{2s-r-1} < \frac{2jr}{2s-r-1} = C(y_j).$$

Since $2s-r-1 > 0$ (see the sentence following (8)),

$$2is - i + 2r < 2jr.$$

Dividing by 2 and noting that $i < r$, we obtain $is < jr$.

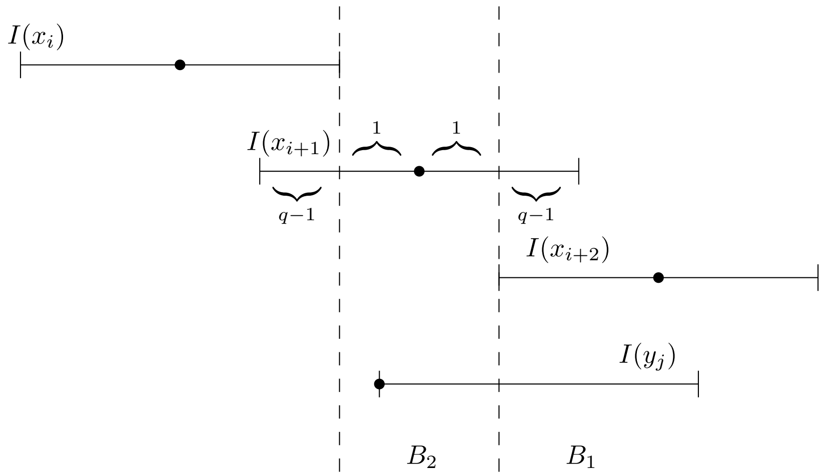

Figure 5: Case $x_i \prec y_j$. If $L(y_j) \in B_2$ and $g(x_i) + s > g(y_j)$, slide $L(y_j)$ and $C(y_j)$ to the left to meet $R(x_i)$.

Since $i \leq r - 2$, we know x_{i+2} is defined. There are now two subcases to consider depending upon whether or not the left endpoint of $I(y_j)$ lies to the right of the left endpoint of $I(x_{i+2})$. These are illustrated in Figure 5 by the regions B_1, B_2.

Subcase B_1**.** Suppose $L(y_j) \geq L(x_{i+2})$, i.e., the left endpoint of $I(y_j)$ is in the interval $B_1 = [L(x_{i+2}), L(y_{s-1})]$. By (12),

$$\frac{2jr}{2s-r-1} \geq \frac{2(i+2)s-(i+2)}{2s-r-1} = \frac{(i+2)(2s-1)}{2s-r-1}.$$

Thus

$$(i+2)(2s-1) \leq 2jr,$$

i.e.,

$$(2is+2s)+(2s-i-2) \leq 2jr.$$

Now $i < r \leq 2(s-1)$ implies that $2s - i - 2 > 0$, so

$$g(x_i) + s = is + s < jr = g(y_j).$$

Subcase B_2**.** Now suppose $L(y_j) < L(x_{i+2})$, as illustrated in Figure 5. Since $R(x_i) < C(y_j) = L(y_j)$, the left endpoint of $I(y_j)$ is in the interval $B_2 = (R(x_i), L(x_{i+2}))$. If $g(x_i) + s \leq g(y_j)$, we are done with this subcase. Otherwise, slide the left endpoint and splitting point of $I(y_j)$ to the left until they meet the right endpoint of $I(x_i)$, i.e., replace $I(y_j)$ by the interval $I'(y_j)$ with $L'(y_j) = C'(y_j) = i(q+1) + 2q$ and $R'(y_j) = R(y_j)$.

We continue sweeping from right to left until we have considered each x_i in turn and modified the y-intervals in this way as needed. All other intervals in the representation are unchanged, i.e., for all other $u \in V, I'(u) = [L'(u), R'(u)] =$

$I(u)$. Also, the labeling of all elements of V is unchanged. This defines a new poset $P' = (V, \prec')$ where $\prec'$ is defined as in Definition 1.

We need to determine which relations in P can change in moving to P'. Since $C'(y_j) = R'(x_i)$ when $I(y_j)$ is modified, the corresponding relation $x_i \prec y_j$ becomes $x_i \parallel' y_j$. We will show these are the only relations that change.

First, by (9), the length of B_2 is

$$L(x_{i+2}) - R(x_i) = (i+2)(q+1) - (i(q+1) + 2q) = 2.$$

Next, the intervals B_2 are disjoint from one another for different x_i because when we compare them for i and $i-1$ we find

$$R(x_i) - L(x_{i+1}) = i(q+1) + 2q - (i+1)(q+1) = q - 1 > 0.$$

Also, the length of each y-interval before modification is $2q > 2$ and modifying it extends it only as far as the left endpoint of the corresponding B_2. So for each x_i at most one y_j can fall into this subcase, and each y_j falls into it for at most one x_i.

Remark 17 *Since the open interval B_2 does not contain the right endpoint of any x-interval, if $R(x_k) \leq L(y_j)$ then $R'(x_k) \leq L'(y_j)$. That is, if we move the left endpoint of a y-interval it does not pass the right endpoint of any x-interval.*

Suppose we have modified $I(y_j)$ for some x_i. The only intervals whose endpoints or splitting points lie in $B_2 \cup \{R(x_i)\} = [R(x_i), L(x_{i+2}))$ are $I(x_i), I(x_{i+1}), I(y_j)$, and if $j \geq 2$ also $I(y_{j-1})$. So the only other relations that could change involve x_{i+1} or y_{j-1} together with y_j.

Before the move $y_j \parallel x_{i+1}$, since (12) implies

$$L(x_{i+1}) < R(x_i) < C(y_j) = L(y_j) < L(x_{i+2}) < R(x_{i+1}),$$

i.e., $C(y_j) \in I(x_{i+1})$. After the move $y_j \parallel' x_{i+1}$, since

$$L'(x_{i+1}) = L(x_{i+1}) < R(x_i) = C'(y_j) < R(x_{i+1}) = R'(x_{i+1}),$$

i.e., $C'(y_j) \in I'(x_{i+1})$.

Now let $j \geq 2$, so that y_{j-1} is defined. Before the move, $y_{j-1} \parallel y_j$. The splitting point $C(y_j) = R(y_{j-1})$ slides to the left at most 2 units but the length of $I(y_{j-1})$ is greater than 2, so after the move

$$L'(y_{j-1}) < C'(y_j) < R'(y_{j-1}).$$

Thus $C'(y_j) \in I'(y_{j-1})$ and so $y_{j-1} \parallel' y_j$.

So there is only one kind of new relation in P', namely, $x_i \parallel' y_j$. We must verify constraint (ii) for this new incomparability. We have $g(x_i) + s > g(y_j)$ by assumption and $g(x_i) < g(y_j)$ by (14). Thus,

$$-r < 0 < g(y_j) - g(x_i) < s \leq r.$$

Because the labeling has not changed, constraints (i) and (ii) remain valid for all the pairs we have considered in this and the previous cases. Also, the forcing cycle C in P remains a forcing cycle in P', with the same relations between consecutive elements and thus the same values of $\mathrm{up}(C)$ and $\mathrm{side}(C)$.

If we redefined any intervals then we only changed the lengths of y-intervals, so in this case the interval representation of P' is no longer unit. However we now argue that it is a proper representation. It suffices to show that none of the new intervals $I'(y_j)$ properly contains any of the other representing intervals. Since we do not shift $L(y_j)$ beyond $L(y_{j-1})$ and $L'(y_{j-1}) \le L(y_{j-1})$, $I'(y_j)$ cannot properly contain any y-interval in P'. Let $k \ge i+2$. Since $L(y_j) < L(x_{i+2})$, we have $R'(y_j) = R(y_j) < R(x_{i+2}) \le R(x_k)$. Thus $I'(y_j)$ cannot properly contain $I'(x_k) = I(x_k)$. Similarly, for $k \le i+1$, we have $L(x_k) \le L(x_{i+1}) < R(x_i) = L'(y_j)$. So $I'(y_j)$ cannot properly contain $I(x_k)$ for any value of k. Thus, the resulting representation is proper and so P' is a split semiorder.

Case $y_j \prec x_i$. For simplicity, we now let $P = (V, \prec)$ denote the poset obtained at the end of the preceding case, i.e., the split semiorder given by a proper representation.

We again sweep through the x-intervals from right to left. For a given x_i, suppose $y_j \prec x_i$ for some y_j. Either we will prove that (i) is true or else we will redefine $C(x_i)$ by moving it to the left. As before, for each x_i this will be the only relation that can change.

We first note that $i \ge 1$, i.e., this case cannot occur with the leftmost x-interval $I(x_0)$. This follows because for each y_j we have $C(y_j) \ge R(x_0) > L(x_0)$, i.e., $y_j \not\prec x_0$.

We next show

$$g(y_j) = jr < is = g(x_i). \tag{15}$$

Since $R(y_j) < C(x_i)$ and these points were not modified in the preceding case, it follows from (12) that

$$\frac{2jr + 2r}{2s - r - 1} < \frac{2is - i + r}{2s - r - 1}.$$

Thus

$$jr + r < is - \frac{i - r}{2},$$

and so

$$jr < jr + \frac{r + i}{2} < is.$$

There are once again two subcases to consider depending upon whether or not the right endpoint of $I(y_j)$ lies to the left of the right endpoint of $I(x_{i-1})$. Note that since $i \ge 1$, we know x_{i-1} is defined. As before, we will either prove that the "up" constraint (i) is true or redefine the poset P accordingly. While we omit the picture, this situation can be illustrated in a way analogous to Figure 5.

Subcase D_1. Suppose $R(y_j) \le R(x_{i-1})$, i.e., the right endpoint of $I(y_j)$ is in the interval $D_1 = [R(y_1), R(x_{i-1})]$. By (12),

$$\frac{2jr + 2r}{2s - r - 1} \le \frac{(i-1)(2s-1) + 2r}{2s - r - 1}$$

and thus

$$2jr \le (i-1)(2s-1) = 2is - i - 2s + 1.$$

Since $i \ge 1$ we have $2jr + 2s \le 2is - i + 1 < 2is + 1$, and since both r and s are integers this implies $2jr + 2s \le 2is$. Therefore

$$g(y_j) + s \le g(x_i),$$

and (i) is true for this subcase.

Subcase D_2. Now suppose $R(y_j) > R(x_{i-1})$. Since $R(y_j) < C(x_i)$, the right endpoint of $I(y_j)$ is in the interval $D_2 = (R(x_{i-1}), C(x_i))$. If $g(y_j) + s \le g(x_i)$, we are done with this subcase. Otherwise, redefine $C(x_i)$ by sliding it to the left to equal $R(y_j)$, i.e., let $C'(x_i) = R(y_j)$. We continue sweeping from right to left, taking each x_i in turn and moving the splitting points $C(x_i)$ as needed. All endpoints and labels remain unchanged. This defines a new poset $P' = (V, \prec')$. We will prove P' has the properties sought in Theorem 16.

For the relation $\prec'$ to define P' as a split semiorder, we must verify that $C'(x_i) \in I'(x_i)$ for each x_i. First note that since $q > 1$, the assumptions of this subcase and (12) imply that $C(x_i) = L(x_i) + q$ is moved to the left by less than $|D_2| = C(x_i) - R(x_{i-1}) = 1 < q$ and so is still in $I(x_i)$. That is, after the shift we have $C'(x_i) \in I'(x_i)$.

Since the representation in the preceding case was proper and only the splitting points in the intervals changed, this representation is also proper and P' is a split semiorder. Since $C'(x_i) \in I'(y_j)$, we know $y_j \parallel' x_i$. We will show these are the only relations that change in moving to P'.

By (12), the intervals D_2 are disjoint from one another for different x_i. The right endpoints of consecutive y-intervals are $2q > 2$ units apart whether or not the left endpoints were changed in the preceding case. So for a given x_i at most one y_j can fall into this subcase. Also, modifying $C(x_i)$ extends it only as far as the left endpoint of the corresponding D_2, so each y_j falls into this subcase for at most one x_i.

Let some $C(x_i)$ be modified. The only intervals whose endpoints or splitting points lie in $D_2 \cup \{R(x_{i-1})\} = [R(x_{i-1}), C(x_i))$ are $I(x_{i-1}), I(x_i), I(y_j)$, and if $j \le s-2$ also $I(y_{j+1})$. So the only other relations that could change involve x_{i-1} or y_{j+1} together with x_i.

Since $i \ge 1$, x_{i-1} is defined and $x_{i-1} \prec x_i$ before the move. The splitting point $C(x_i)$ slides to the left but not as far as the right endpoint of $I(x_{i-1})$, so after the move $x_{i-1} \prec' x_i$.

Now let $j \le s-2$, so that y_{j+1} is defined. Whether or not we modified $L(y_{j+1})$ in Case ($\prec xy$), Remark 17 and (12) imply that in the current case

$$R(x_{i-1}) \le L(y_{j+1}) \le R(y_j) < C(x_i) = R(x_{i-1}) + 1 \le L(y_{j+1}) + 1 < R(y_{j+1}).$$

Thus $C(x_i) \in I(y_{j+1})$ and so $x_i \parallel y_{j+1}$ before the move. Modifying $C(x_i)$ only slides it as far as $R(y_j)$, so after the move $C'(x_i) \in I'(y_{j+1})$ and $x_i \parallel' y_{j+1}$.

Therefore, the only changes in the partial ordering can be from $y_j \prec x_i$ to $y_j \parallel' x_i$.

Next, we must prove (ii) holds for $y_j \parallel' x_i$. Since $g(y_j) + s > g(x_i)$ by the assumptions of this subcase and $g(y_j) < g(x_i)$ by (15), we have

$$-r < 0 < g(x_i) - g(y_j) < s \le r.$$

So the constraints (i) and (ii) hold for all the pairs we have considered in this and the previous cases. The forcing cycle C in P remains a forcing cycle in P'.

Finally, since $\mathrm{up}(C) = r, \mathrm{side}(C) = s$, Theorem 6 implies $wd_F(P') \ge r/s$. On the other hand, the labeling f constructed in the proof shows $wd_F(P') \le r/s$, so we conclude $wd_F(P') = r/s$. This completes the proof of Theorem 16. $\square$

The following example shows that we may indeed need to modify the partial ordering as we did in the final two cases in Theorem 16. Let $r = 7, s = 6$, so that $q = 7/4$. In Case $x_i \prec y_j$ we have $x_5 \prec y_5$, since

$$C(x_5) = \frac{62}{4} < \frac{70}{4} = L(y_5), \quad R(x_5) = \frac{69}{4} < \frac{70}{4} = C(y_5).$$

When $i = 5$, the region $B_2 = (R(x_5), L(x_7)) = (\frac{69}{4}, \frac{77}{4})$ and contains $L(y_5) = \frac{70}{4}$. Since $g(x_5) + s = 36 > 35 = g(y_5)$, we must redefine $L(y_5) = C(y_5)$, sliding them to the left from $\frac{70}{4}$ to $R(x_5) = \frac{69}{4}$. In the modified poset P', we then have $x_5 \parallel' y_5$.

Similarly, in Case $y_j \prec x_i$ we have $y_1 \prec x_2$ and must slide $C(x_2)$ to meet $R(y_1)$. This change creates the relation $y_1 \parallel' x_2$ in P'.

Finally, we can combine Theorems 8 and 16 to describe the range of the fractional weak discrepancy function for split semiorders. We will see that the way in which we represent $wd_F(P)$ as $wd_F(P) = q = r/s$ determines whether there is an optimal forcing cycle C with $r = \mathrm{up}(C)$ and $s = \mathrm{side}(C)$.

Corollary 18 *For any rational number $q > 0$, there exists a split semiorder P with $wd_F(P) = q$ if and only if q can be written as $q = r/s$ for some integers r, s with $0 \le s - 1 \le r < 2s$.*

Proof. First, suppose P is a split semiorder with $wd_F(P) = q$. We must show $q = r/s$ for some r, s as stated in the theorem. If $q = 0$ we let $r = 0,\ s = 1$. If $0 < q < 1$, then Theorem 15 implies P is a semiorder and $q = \frac{r}{r+1}$ for some integer $r \ge 1$. So we can let $s = r + 1$ and then $1 \le s - 1 = r < 2(s-1) < 2s$.

Now suppose $q \ge 1$. Since P has an incomparable pair, Theorem 6 implies it has an optimal forcing cycle C. Let $r = \mathrm{up}(C)$ and $s = \mathrm{side}(C)$. Then $2 \le s \le r$ and, by Theorem 8, $r \le 2(s-1)$. Thus $1 \le s - 1 < r \le 2(s-1) < 2s$. So in all cases q has the desired representation.

Conversely, suppose $q = r/s$, where $0 \le s - 1 \le r < 2s$. We must produce an appropriate split semiorder P. If $s = 1$ and $q = r = 0$, we can let P be any

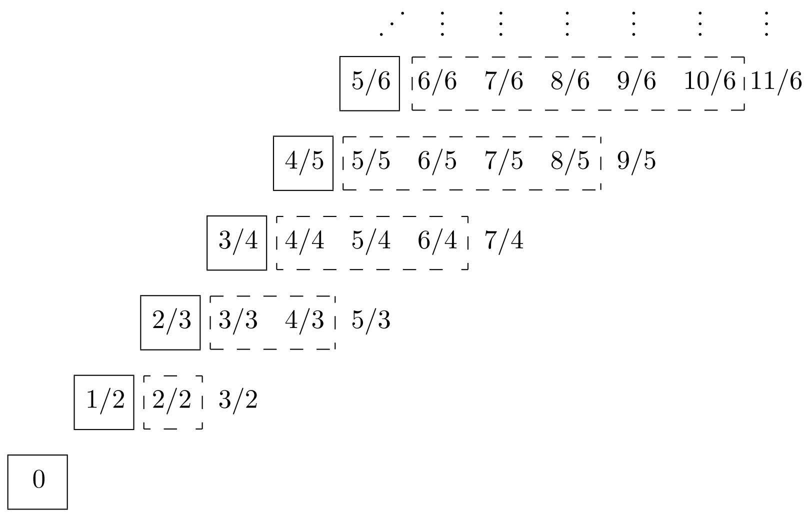

Figure 6: The range of values taken by wd_F for split semiorders.

linear order. If $s = 1$ and $q = r = 1$, we can let $P = \mathbf{3} + \mathbf{1}$, which is a split semiorder (see Figure 1) and has $wd_F(P) = 1$.

Now let $s \geq 2$. First consider the case in which $r \leq 2(s-1)$. Then by Theorem 16 there is a split semiorder P with $wd_F(P) = q$ and having an optimal forcing cycle C with $\mathrm{up}(C) = r$, $\mathrm{side}(C) = s$. Now consider the case where $r > 2(s-1)$. Then $r = 2s - 1$ and by Theorem 8 there is no split semiorder with such a forcing cycle. In this case we let $r' = 2r$, $s' = 2s$. We will show that r', s' satisfy the hypotheses of Theorem 16. We have $r \geq s$, since otherwise $r = 2s - 1 < s$ implies $s = 0$. So $2s - 1 \leq 2r - 1 < 2r = 2(2s-1)$. Thus $s' - 1 < r' = 2(s'-1)$. Now by Theorem 16 there is a split semiorder P with $wd_F(P) = r'/s' = q$ and having an optimal forcing cycle C with $\mathrm{up}(C) = r'$, $\mathrm{side}(C) = s'$. □

Corollary 18 can be used to extend the scope of Theorems 8 and 16. For example, by Theorem 8 there is no split semiorder P with $wd_F(P) = 3/2$ that has an optimal forcing cycle C with $r = \mathrm{up}(C) = 3$, $s = \mathrm{side}(C) = 2$. But by Corollary 18 there is a split semiorder P with $wd_F(P) = 3/2$ having an optimal forcing cycle C with $r' = \mathrm{up}(C) = 6$ and $s' = \mathrm{side}(C) = 4$. In fact, Figures 3 and 4 gave an example of such a split semiorder.

Figure 6 illustrates Corollary 18. The solid boxes show the range of wd_F for semiorders. The dashed boxes show the r-s pairs ($r \geq 1, s \geq 2$) for which there is a split semiorder P that is not a semiorder and has an optimal forcing cycle C with $\mathrm{up}(C) = r, \mathrm{side}(C) = s$. For the unboxed pairs, $wd_F(P) = r/s$ and there is an optimal forcing cycle with $\mathrm{up}(C) = 2r, \mathrm{side}(C) = 2s$.

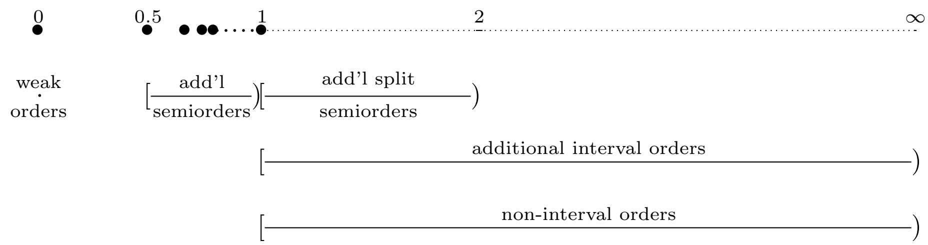

Figure 7: The range of wd_F for various classes of posets.

4 Conclusion.

In this section we place our results on the range of the fractional weak discrepancy function for split semiorders in the context of earlier results about the range for other classes of posets.

Linear orders have no incomparable pairs, so for them $wd_F(P) = 0$. For nonlinear orders, Theorem 6 implies that $wd_F(P)$ is always a rational number. The simplest case is that of the weak orders, which include the linear orders: $wd_F(P) = 0$ if and only if P is a weak order [5]. Theorem 15, proved in [6], describes the range of the wd_F function over the semiorders, which include the weak orders. In particular, $\{wd_F(P) : P \text{ a semiorder}\} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$. Since every semiorder is also a split semiorder, this set is contained in the range of wd_F over all split semiorders. This is also the case for interval orders, since each semiorder is an interval order.

So $wd_F(P) \geq 1$ for any P that is not a semiorder. Corollary 18 shows that the additional values of $wd_F(P)$ that occur when P is a split semiorder but not a semiorder are all the rational numbers in $[1, 2)$.

Moreover, each rational $q \geq 1$ is the fractional weak discrepancy of both an interval order that is not a semiorder and of a poset that is not an interval order [7]. Figure 7 combines this fact with the other results summarized in this section, illustrating the range of wd_F for successively larger classes of posets.

We close with two open questions. What is $wd_F(P)$ for a subsemiorder, i.e., a poset having no induced $\mathbf{4} + \mathbf{1}$ or $\mathbf{3} + \mathbf{2}$? (See the paragraph before Definition 1.) More generally, what is $wd_F(P)$ for an order containing no $\mathbf{r} + \mathbf{s}$ for $r + s = M$, where $M \geq 5$?

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