UNIQUENESS OF PARRY MAPS, AND INVARIANTS FOR TRANSITIVE PIECEWISE MONOTONIC MAPS

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Abstract. Parry showed that every continuous transitive piecewise monotonic map τ of the interval is conjugate (by an order preserving homeomorphism) to a uniformly piecewise linear map T (i.e., one with slopes $\pm s$). In the current article, it is shown that the map T is unique. This is proven by showing that any order-preserving conjugacy between two continuous transitive uniformly piecewise linear maps is the identity map. (This is false for non-transitive maps.) This generalizes analogous results for transitive Markov maps by Block and Coven.

Analogous results hold for discontinuous transitive piecewise monotonic maps if the maps have positive entropy, but fail in the zero entropy case.

A map $\tau : [0,1] \to [0,1]$ is *piecewise monotonic* if there is a partition $0 = a_0 < a_1 < \ldots < a_n = 1$ such that τ is continuous and strictly monotonic on each interval (a_{i-1}, a_i) for $1 \le i \le n$. An example of such a map is a *uniformly piecewise linear* map, i.e., a map τ for which there is a partition $0 = a_0 < a_1 < \ldots < a_n = 1$ such that τ is linear on (a_{i-1}, a_i) for $1 \le i \le n$, with slope $\pm s$ for some $s > 0$. Throughout this article, we write I for the interval $[0,1]$, and $\tau : I \to I$ will be a piecewise monotonic map of the unit interval, not necessarily continuous unless that is stated..

It is well known that the quadratic map given by $\tau(x) = 4x(1-x)$ is transitive, and is conjugate to the uniformly piecewise linear map $T(x) = 1 - |1 - 2x|$. Parry [6] showed that every (strongly) transitive piecewise monotonic map τ is conjugate to a uniformly piecewise linear map T, cf. Theorem 2.

The map T is not unique, since two or more uniformly piecewise linear maps can be conjugate. For example, the map $\phi(x) = 1 - x$ is a homeomorphism that conjugates any uniformly piecewise linear map T onto another uniformly piecewise linear map $\phi \circ T \circ \phi^{-1}$, which we call the *reflection* of T, cf. Definition 6. However, we will show that if τ is a continuous transitive piecewise monotonic map, this is the extent of the ambiguity: two conjugate continuous transitive uniformly piecewise linear maps are either the same or are reflections of each other, cf. Theorem 10. (This is false without the assumption of transitivity, cf. Example 13.) Thus for a continuous transitive piecewise monotonic map, there is an order preserving homeomorphism onto a unique uniformly piecewise linear map T. We will prove analogous results for discontinuous τ with the added hypothesis of positive entropy.

For continuous τ, uniqueness of this "Parry map" T (up to reflection) was proven with the additional assumption that τ was Markov by Block and Coven [1].

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1. Uniqueness of Parry maps

Definition 1. A map $\tau : I \to I$ is *transitive* if for each pair U, V of non-empty open subsets of I, there exists $n \geq 0$ such that $\tau^n(U) \cap V \neq \emptyset$.

The following is the result of Parry mentioned in the introduction. Here h_τ denotes the topological entropy of τ.

Theorem 2 (Parry [6])**.** *If a piecewise monotonic map $\tau : I \to I$ is transitive, then there is an increasing conjugacy from τ onto a uniformly piecewise linear map with slopes $\pm s$, where $s = \exp(h_\tau) \geq 1$. If τ is continuous, then $s > 1$.*

Proof. The map τ is said to be *strongly transitive* if for every open interval $J \subset I$, a finite number of iterates of J cover I. Every transitive piecewise monotonic interval map is strongly transitive, cf. [7, Theorem 2.5] for the continuous case, and [8, Prop. 2.9] for the discontinuous case. Every strongly transitive piecewise monotonic interval map τ is conjugate to a uniformly piecewise linear map T with slopes $\pm s$, where $s = \exp(h_\tau)$, cf. [6] and [9, Cor. 4.4]. Since T must be surjective, then $s \geq 1$.

If τ is continuous, then T is also continuous as well as surjective, so $s = 1$ would force $T(x) = \pm x$, which is not transitive. Thus $s > 1$.

□

Definition 3. If $\tau : I \to I$ is piecewise monotonic and transitive, a *Parry map* for τ is a uniformly piecewise linear map T admitting an increasing conjugacy from τ onto T.

Definition 4. A probability measure μ on I is *scaled by τ by a factor s* if $\mu(\tau(E)) = s\mu(E)$ for all Borel sets E on which τ is injective. In that case, we also say μ is a *scaling measure* for τ.

Such measures are a special case of the notion of *conformal measures*, due to Denker and Urbański [2]. Scaled measures play a key role in Parry's paper [6], though the term "scaled measure" isn't used there. In order to prove the result in Theorem 2, Parry proved the existence of a measure μ scaled by τ. This measure μ then can be used to construct a conjugacy onto a uniformly piecewise linear map T with slopes $\pm s$ by defining $h(x) = \mu([0, x))$ for $x \in I$, and $T = h \circ \tau \circ h^{-1}$. In fact, there is a 1-1 correspondence of conjugacies of τ onto uniformly piecewise linear maps and measures scaled by τ, as we now show.

If $h : X \to Y$ is bijective with h and h^{-1} Borel maps, and μ is a Borel measure on Y, then we write $\mu \circ h$ for the Borel measure on X given by $(\mu \circ h)(E) = \mu(h(E))$ for Borel sets E.

Parts of the following result appear frequently in proofs of conjugacy involving uniformly piecewise linear maps, so it may be considered to be folklore.

Theorem 5. *Let μ be a non-atomic probability measure μ on I of full support, and suppose a piecewise monotonic map $\tau : I \to I$ scales μ by a factor $s > 0$. Define $h_\mu : I \to I$ by $h_\mu(x) = \mu([0, x])$. Then h_μ is a homeomorphism from I onto I, and the map $T = h_\mu \circ \tau \circ h_\mu^{-1}$ is piecewise linear with slopes $\pm s$. Furthermore, the map $\mu \mapsto h_\mu$ is a bijection from the set of non-atomic probability measure μ on I of full support, scaled by τ, onto the set of increasing conjugacies h from τ onto uniformly piecewise linear maps. The inverse of the map $\mu \mapsto h_\mu$ is $h \mapsto m \circ h$, with m being Lebesgue measure.*

Proof. It is straightforward to check that h_μ is a homeomorphism and that $h_\mu \circ T \circ h_\mu^{-1}$ is uniformly piecewise linear with slopes $\pm s$, cf., e.g., [9, Prop. 3.6].

Now suppose that $h : I \to I$ is any increasing conjugacy from τ onto a uniformly piecewise linear map $T : I \to I$ with slopes $\pm s$, and define $\mu = m \circ h$. Then μ is a non-atomic measure on I of full support, scaled by τ by the factor s. For each $x \in I$, we have $h_\mu(x) = \mu([0, x]) = m([0, h(x)] = h(x)$, so $h_\mu = h$.

Now let μ denote any non-atomic probability measure on I of full support, scaled by τ by a factor s. Then for each $x \in I$,

$$(m \circ h_\mu)([0, x]) = m([0, h_\mu(x)] = h_\mu(x) = \mu([0, x]).$$

Thus $m \circ h_\mu = \mu$, which completes the proof that the maps $\mu \mapsto h_\mu$ and $h \mapsto m \circ h$ are mutual inverses. □

Definition 6. If $T : I \to I$ and $\phi(x) = 1 - x$, then $\widetilde{T} = \phi \circ T \circ \phi^{-1}$ is the *reflection* of T. (To motivate this terminology, we observe that the graph of $\widetilde{T}$ is the graph of T reflected in the point $(1/2, 1/2)$.)

Definition 7. A piecewise monotonic map $\tau : I \to I$ is *essentially injective* if there are no intervals J_1 and J_2 such that $\tau(J_1) = \tau(J_2)$.

(Throughout this article, an interval is a connected subset of I with more than one point.) A piecewise monotonic map is essentially injective iff it is injective on the complement of a finite set.

Lemma 8. *Let $T : I \to I$ be transitive and uniformly piecewise linear, with slopes $\pm s$ with $s \geq 1$. Then the following are equivalent.*

(1) $s = 1$
(2) *T has zero topological entropy.*
(3) *T is essentially injective*

Proof. (1) $\iff$ (2) For a uniformly piecewise linear map τ with slopes $\pm s$, with $s \geq 1$, the topological entropy is $h_\tau = \ln s$, cf. [5] for the continuous case, and [9, Prop. 3.7] in general. Thus (1) and (2) are equivalent

(1) $\iff$ (3) If $I_1, \ldots, I_n$ are the intervals of monotonicity of T, the total length of $T(I_1), \ldots, T(I_n)$ will be s, and by transitivity, the union of these intervals must cover all of I except perhaps for a finite set of points. These intervals will overlap only at endpoints iff the sum of their lengths is 1, i.e., iff $s = 1$.

□

The key to the uniqueness result we're after is the following uniqueness result for scaled measures, cf. [9, Cor. 4.6].

Theorem 9. *If $\tau : I \to I$ is piecewise monotonic, transitive, and not essentially injective, then the probability measure μ scaled by τ is unique.*

Now we can prove a kind of rigidity for uniformly piecewise linear transitive maps.

Theorem 10. *If $T_1 : I \to I$ and $T_2 : I \to I$ are transitive and uniformly piecewise linear, with slopes $\pm s$ with $s > 1$, and h is an increasing conjugacy from T_1 to T_2, then h is the identity map.*

Proof. By Theorem 9, Lebesgue measure m is the unique scaling measure for T_1 and T_2. On the other hand, $E \mapsto m(h(E))$ is also a scaling measure for T_1, so $m(h(E)) = m(E)$ for all Borel sets E. Since h is increasing, then $h(0) = 0$, so for all x,

$$x = m([0, x]) = m(h([0, x]) = m([0, h(x)]) = h(x).$$

□

Theorem 11. *If $\tau : I \to I$ is transitive and piecewise monotonic, and is not essentially injective, then the Parry map associated with τ is unique.*

Proof. If T_1 and T_2 were two Parry maps for τ, then T_1 and T_2 would not be essentially injective, and there would be an increasing conjugacy from T_1 to T_2. This conjugacy must be the identity map by Lemma 8 and Theorem 10. □

Corollary 12. *If $\tau : I \to I$ is transitive and piecewise monotonic, and is either continuous or has positive topological entropy, then the Parry map associated with τ is unique.*

Proof. Assume $\tau : I \to I$ is transitive and piecewise monotonic.

If τ also is continuous and were essentially injective, then in fact τ would be injective, so couldn't be transitive. Thus τ is not essentially injective.

If instead we assume τ has positive topological entropy, then by Theorem 2, τ is conjugate to a uniformly piecewise linear map T with slopes $\pm s$ with $s = \exp(h_\tau) > 1$. By Lemma 8, τ is not essentially injective.

Now the corollary follows from Theorem 11. □

2. Non-transitive piecewise monotonic maps

Example 13. Transitivity is not redundant in Theorem 9, Theorem 11, or Corollary 12. To see this, let T be the map which is linear on the intervals $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{3}{4}]$, $[\frac{3}{4}, 1]$ and satisfies $T(0) = 1/2$, $T(1/4) = 0$, $T(3/4) = 1$, $T(1) = 1/2$. Fix a with $0 < a < 1$, and define a homeomorphism $h_a : I \to I$ by setting

$$h_a(x) = \begin{cases} 2ax & \text{if } 0 \le x < 1/2 \\ 2a - 1 + 2(1-a)x & \text{if } 1/2 \le x \le 1 \end{cases}$$

(For motivation, let m be Lebesgue measure, and define μ_1 and μ_2 to be the probability measures given by $\mu_1(E) = 2m(E \cap [0, 1/2])$ and $\mu_2(E) = 2m(E \cap [1/2, 1])$. Then μ_1 and μ_2 are scaled by T by the factor 2. If we define $\mu_a = a\mu_1 + (1-a)\mu_2$, then for each a in $(0, 1)$, μ_a is scaled by T by the factor 2, and h_a is the conjugacy corresponding to the scaling measure μ_a as described in Theorem 5.)

Define $T_a = h_a \circ T \circ h_a^{-1}$. By Theorem 5, T_a is uniformly piecewise linear. Observe that T equals its own reflection, and for $a \ne 1/2$, T_a is not equal to T. (Indeed, if $T_a = T$, then $h_a \circ T = T \circ h_a$, but $h_a(T(0)) = h_a(1/2) = a$, while $T(h_a(0)) = T(0) = 1/2$.) Thus for $a \ne 1/2$, T and T_a are uniformly piecewise linear maps that are conjugate but are neither equal nor reflections.

3. The zero entropy case

By Lemma 8, $s = 1$ is the zero entropy case for transitive uniformly piecewise linear maps. Well known examples of such maps are *interval exchange maps*, which are bijective piecewise linear maps $\tau : [0, 1) \to [0, 1)$ with pieces that are linear with slope 1, with the map right continuous at each point of discontinuity. We extend such a map to $[0,1]$ by defining it so that it is left continuous at 1. We will see that it is possible to have two distinct conjugate transitive interval exchange maps, so Theorem 10 cannot be extended to the zero entropy case, and it will follow that uniqueness of the Parry map (Theorem 11) doesn't hold without the assumption that τ is essentially injective.

An interval exchange map T is *minimal* if the (two-sided) orbit of every point in $[0,1)$ is dense. The argument in the proof of [3, first ttheorem] shows that for a minimal interval exchange map T and any open interval J, a finite number of iterates of J cover $[0,1)$, so T is transitive viewed as a map on $[0,1]$.

Example 14. Let T_1 be a minimal interval exchange map admitting an invariant probability measure other than Lebesgue measure. (An example of such a map was given in [4].) Let μ be a non-trivial convex combination of Lebesgue measure and another invariant probability measure. Then μ has full support. By minimality of T_1, orbits of every point are infinite, so any invariant probability measure is necessarily non-atomic. Define $h(x) = \mu([0,x])$. Then h will be a non-trivial homeomorphism from I onto I. Furthermore, the invariance of μ implies that $T_2 = h \circ T_1 \circ h^{-1}$ is uniformly piecewise linear with slopes ± 1.

Suppose $T_2 = T_1$. Then $T_1 = h \circ T_1 \circ h^{-1}$, so $T_1 \circ h = h \circ T_1$. Thus $T_1^n(h(0)) = h(T_1^n(0))$ for all n, and $h(0) = 0$, so $T_1^n(0) = h(T_1^n(0))$. Hence h fixes the orbit of 0, which is dense by minimality. Thus h is the identity, a contradiction.

Hence T_1, T_2 are distinct transitive uniformly piecewise linear maps and h is an increasing conjugacy from T_1 onto T_2, showing that Theorem 10 doesn't hold if $s = 1$.

4. Parry maps as invariants

For each continuous transitive piecewise monotonic map τ, we can consider the unique associated Parry map T as a complete invariant (with respect to order preserving conjugacies). (Alternatively, to get a conjugacy invariant we could use the unordered pair consisting of T and its reflection.)

Of course, we could express T as a numeric invariant. For example, if T is continuous, then it is determined by $T(0)$, the slope s, whether T increases or decreases on the first interval of monotonicity, and the lengths of the intervals of monotonicity.

These invariants are not independent — for example, since T is surjective, the value of $T(0)$ is determined by the remaining pieces of information, and the lengths of the intervals of monotonicity must add up to 1.

If $\tau : I \to I$ is continuous, transitive and piecewise monotonic, let μ be the unique scaling measure, cf. Theorem 9, and h the associated conjugacy onto the Parry map with slopes $\pm s$. Define a metric on I by $d(x,y) = h(y) - h(x) = \mu([x,y])$ for $y \geq x$. With respect to this metric, we can interpret s as the rate at which τ-orbits separate. Similarly, the lengths of the intervals of linearity of T can be thought of as intrinsic measures of the length of each interval of monotonicity of τ.

What are the possible invariants? In the case of unimodal maps T, this is easy to determine. By reflecting T if necessary, we may assume that T increases and then decreases. By surjectivity, either $T(0) = 0$ or $T(1) = 0$; by transitivity the latter must hold. If the slopes are $\pm s$, then one finds that the critical point is $c = 1 - 1/s$, and so $T(0) = 2 - s$. It is well-known that this map is transitive iff $\sqrt{2} \leq s \leq 2$, cf., e.g., [9, Lemma 8.1]. Thus for such maps, the slope (or equivalently, the topological entropy) is a complete invariant. A description of the transitive bimodal uniformly piecewise linear maps can be found in [10].

A similar analysis applies to transitive discontinuous piecewise monotonic maps, except that we must exclude the essentially injective case, and to describe the map T we also must specify whether the map increases or decreases on each interval of monotonicity, and the size and sign of jumps at each endpoint of an interval of monotonicity.

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