

Below are some thoughts on the topics we're going to talk about in section today. I'm leaning towards discussing the philosophy behind the concepts so that you understand not just what we're talking about, but why.

1 Nullspace and Columnspace

A member of our section asked me two excellent questions after last week. The first was "Isn't the nullspace of a matrix the same as what we've been doing before in solving systems of equations?" The great news about nullspace and columnspace is that they are very much like the kinds of problems we've done already. In fact, to compute the null or columnspace of a matrix we even follow the same procedures we use to follow in solving a system of equations (row reducing an appropriate matrix). This is good news! Up to this point in the book, you essentially need only one tool in your toolbox (namely, the ability to row reduce a matrix) and you can answer almost any question they throw at you. Here are some examples of questions you can answer by row reducing matrices:

- What are the solutions to $A\vec{x} = \vec{0}$? What about $A\vec{x} = \vec{b}$ for some $\vec{b} \neq \vec{0}$?
- How many solutions are there to $A\vec{x} = \vec{b}$? What rules do these solutions obey?
- Are the columns of A linearly independent or not?
- What are the linear relations among the columns of A ?
- What vectors are orthogonal to the rows of A ?

The second question she asked was "Why go through the bother of giving nullspace and columnspace a name if they are essentially the same as what we've already done?" So far in this class we've really spent most of our time talking about *algebra* (that is, talking about algebraic structures like systems of linear equations or algebraic operations like elementary row operations). But linear algebra is about to start looking a lot more geometric, and this geometry really starts when we introduce the concept of subspaces. The prototypes of subspaces are our old friends nullspace and columnspace, and so by defining the concept of nullspace and columnspace we're attempting to connection the algebra you already know with the geometry we're about study. Exciting!

2 Subspaces

Like I said in the previous section, I think of subspaces as capturing the geometry of linear algebra. But for now, you should focus on *precisely* what subspaces are. In particular you should write the following definition on your heart.

Definition 2.1 A collection of vectors $V \subseteq \mathbb{R}^n$ is called a subspace of \mathbb{R}^n if it satisfies the following rules

- $\vec{0} \in V$;
- for any $\vec{v}, \vec{w} \in V$, the vector $\vec{v} + \vec{w}$ is in V ; and
- for any $\vec{v} \in V$ and $c \in \mathbb{R}$, the vector $c\vec{v}$ is in V .

In plain terms, a subspace is a collection of vectors which contain $\vec{0}$ and are closed under addition and scalar multiplication (we say ‘closed under addition’ because addition of vectors in V stays in V , and ditto for ‘close under scalar multiplication’). An important question is this: “Somebody has given me a collection of vectors V . How do I show that V is a subspace?” The answer isn’t so bad. All you have to do to show that V is a subspace is verify that it satisfies all the properties of a subspace. For instance, does your collection contain the vector $\vec{0}$? If you add any two vectors in V , will the sum be in V ? If you scale any vector in V , is the scaled vector in V ? If you can answer all these questions with an affirmative, you have yourself a subspace. Otherwise, you don’t.

Subspaces connect to the stuff we’ve already learned in a very nice way: the collections $N(A)$ and $C(A)$ associated to a matrix A are, in fact, subspaces. Yes!

3 Bases

A subspace is a wonderful thing in a lot of ways, but one way that makes it kind of scary is that it (almost always, at least) will have lots and lots and lots (and lots) of vectors in it. In fact, almost all subspaces will have infinitely many elements (which ones will have a finite number of elements?). In order to understand a subspace, however, we would like to have a convenient way to describe what its vectors look like. For instance, it’s a lot better to know your subspace takes the form

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 = x_2 \text{ and } x_3 = 0 \right\}$$

or you might prefer to describe (the same) subspace as

$$\text{span} \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right).$$

In general one will usually describe a subspace V as the span of vectors $\vec{v}_1, \dots, \vec{v}_d$ in V which

- are linearly independent
- span all of V .

The second condition ensures that the collection of vectors has ‘enough’ information to describe V , while the first ensures that the collection doesn’t have ‘redundant’ information. In fact, the conditions above are exactly the conditions a collection of vectors must satisfy in order to be a **basis** of V . To reiterate: a basis is a collection of vectors linearly independent, spanning vectors for V . The power of have a basis is that you have a collection of vectors which simultaneously have these two desirable properties, and you need to remember that a basis has both of these properties when want to use them to prove stuff.

One natural question is “How do I show that vectors $\vec{v}_1, \dots, \vec{v}_d$ is a basis for a space V ?” Well, your collection needs to be a collection of vectors in V , and they must be both linearly independent and span V . If your vectors satisfy these conditions, they are a basis. If they don’t, they are not a basis.

Another natural question is “How do I compute a basis if I’m given a subspace V ?” I’ll give you a partial answer: to find bases for $N(A)$ and $C(A)$, you should row reduce the matrix A . Pivot columns correspond to columns of A which form a basis for $C(A)$, and non-pivot columns correspond to basis elements of $N(A)$.

Warning. A subspace does not have just one basis. In general, a subspace V will have lots and lots and lots (and lots) of potential bases. So just because you and Joe Schmo get different bases for a subspace V doesn’t mean you are both wrong.

4 Dimension

I said in the previous section that a given subspace can have lots and lots of different bases. The tie that binds these bases is that they must all have the same number of elements. This is critically important for you to remember. If you compute a basis of a subspace V and Joe Schmo computes a basis of V , then you must both have the same number of vectors in your bases, even if your bases are otherwise different.

Because of this nice result the following definition makes sense

Definition 4.1 *The dimension of a subspace V is the number of elements in a basis of V*

Knowing the dimension of a subspace is really a powerful amount of information for the following reason. Suppose that you are given a subspace V and you are told it has dimension d . Now suppose that you are also given vectors $\vec{v}_1, \dots, \vec{v}_d$ which are in V . How would you check if these vectors are a basis for V ? Well, you should check that these vectors are linearly independent and span all of V (that’s two things to check). However, since you know that V is d -dimensional, and since the collection of vectors you’re holding has d vectors, you can get away with checking *either* that $\vec{v}_1, \dots, \vec{v}_d$ are linearly independent *or* that they span all of V . You don’t have to check both, since the validity of one will imply the other. This is great, because it cuts down the amount of work you have to do in determining whether a collection is a basis.

There are certain subspaces whose dimension get a special name. In particular, the **nullity** of a matrix A is $\dim(N(A))$ and the **rank** of a matrix A is $\dim(C(A))$. There is a beautiful connection between these numbers.

Theorem 4.1 (The rank-nullity theorem) *If A is an $r \times c$ matrix, then*

$$\text{nullity}(A) + \text{rank}(A) = c.$$

This theorem is also called the fundamental theorem of linear algebra, so you know it has to be important.

The rank and nullity of a matrix give you some information about how many solutions you can expect to a system of equations. Recall that in general a system $A\vec{x} = \vec{b}$ has either 0, 1 or infinitely many solutions. If you know the rank and nullity of a matrix, however, you can generally narrow the possibilities down to only two of these three options (but sometimes you can say precisely how many solutions the system will have, just from knowing the rank and nullity!).

5 Exercises

1. Find all vectors orthogonal to $\begin{pmatrix} 2 \\ 3 \\ 7 \\ 11 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$.

2. How would you find all vectors orthogonal a given collection of vectors $\vec{w}_1, \dots, \vec{w}_s$?

3. Suppose that \vec{v}, \vec{w} and \vec{z} are linearly independent vectors. Are the vectors $2\vec{v} - \vec{w} + 5\vec{z}$, $\vec{v} + 11\vec{w} - 3\vec{z}$ and $\vec{w} - \vec{z}$ linearly independent also?

4. Prove that the collection of all vectors orthogonal to $\begin{pmatrix} 2 \\ 3 \\ 7 \\ 11 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$ is subspace.

5. Is the collection of vectors

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 x_2 x_3 = 0 \right\}$$

a subspace?

6. Is the collection of vectors

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : 2x_1 + x_2 - x_3 = 0 \right\}$$

a subspace? What if you change 0 to 1?

7. Find a subspace of \mathbb{R}^5 which does not have infinitely many elements.

8. Is the collection

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix} \right\}$$

a subspace?

9. Is the collection

$$\text{span} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \cup \text{span} \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$$

a subspace?

10. Prove that the intersection of the two planes $x + y + z = 0$ and $x - y + 14z = 0$ is a subspace. Find a basis for this subspace.

11. Find a basis for the nullspace and columnspace of A when

$$A = \begin{pmatrix} 7 & 3 & 9 & 1 \\ 3 & 7 & 1 & 9 \\ 9 & 1 & 13 & -3 \\ 1 & 9 & -3 & 12 \end{pmatrix} \quad \text{and when} \quad A = \begin{pmatrix} -1 & 2 & 4 \\ 2 & -4 & -8 \end{pmatrix}.$$

12. Find a basis for the subspace of all vectors orthogonal to $\begin{pmatrix} 2 \\ 3 \\ 7 \\ 11 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$.
13. Suppose you are told that $\text{nullity}(A) > 0$. How many solutions can $A\vec{x} = \vec{b}$ have?
14. Suppose that A is a 4×6 matrix whose nullity is 1. What is its rank? How many solutions does $A\vec{x} = \vec{0}$ have? How many solutions does $A\vec{x} = \vec{b}$ have when $\vec{b} \neq \vec{0}$?
15. Suppose that A is a matrix whose reduced row echelon form is R . Is a basis for $C(R)$ also a basis for $C(A)$?
16. Suppose that $\vec{v}_1, \dots, \vec{v}_d$ is a basis for $N(A)$ and that $\vec{v} \notin \text{span}(\vec{v}_1, \dots, \vec{v}_d)$. Prove that $A\vec{v} \neq \vec{0}$.
17. Prove that if $V \subseteq W$ are subspaces, then $\dim(V) \leq \dim(W)$.
18. Prove that if $V \subseteq W$ are subspaces with $\dim(V) = \dim(W)$, then $V = W$.
19. Suppose that A has more rows than it has columns. Prove that there is some \vec{b} so that $A\vec{x} = \vec{b}$ has no solutions.
20. For $\vec{v}, \vec{w} \in \mathbb{R}^3$, prove that $\vec{v} \times \vec{w}$ is orthogonal to both \vec{v} and \vec{w} .