A FEW SOLUTIONS

Here are solutions to a couple of problems that I gave in Section. I've chosen to write up a solution to the first problem because I know that giving a proof of when a collection of vectors forms a subspace can be tricky sometimes. I chose to write up a solution to the second problem because I think it's not easy, and therefore a good test of how comfortable you feel with proofs (in fact, I doubt that anyone was able to solve this problem after I gave it in class). I'm doubt that anything like these problems will show up on your test, but I am convinced that if you understand how these problems work you can probably knock down a lot of the more theoretical problems you'll come across.

Problem. Suppose that $W \subseteq \mathbb{R}^r$ is a subspace and that $T : \mathbb{R}^c \to \mathbb{R}^r$ is a linear transformation. Show that

$$T^{-1}(W) = \{\overrightarrow{v} \in \mathbb{R}^c : T(\overrightarrow{v}) \in \mathbb{R}^r\}$$

is a subspace of $\mathbb{R}^r$.

Solution. We need to check the 3 rules that a subspace must obey. First, we have to show that $\overrightarrow{0} \in \mathbb{R}^c$ is an element of $T^{-1}(W)$; unwinding the definition of $T^{-1}(W)$, this means that we need to show $T(\overrightarrow{0}) \in W$. Remember that we proved that $T(\overrightarrow{0}) = \overrightarrow{0}$ since T is a linear operator (the $\overrightarrow{0}$ on the left hand side of the equation is the zero vector in $\mathbb{R}^c$, while the $\overrightarrow{0}$ on the right hand side of the equation is in $\mathbb{R}^r$). Since W is a subspace, $\overrightarrow{0} \in W$ (this $\overrightarrow{0}$ is in $\mathbb{R}^r$, of course, since that's where W lives). Putting these together gives $T(\overrightarrow{0}) = \overrightarrow{0} \in W$, and therefore $\overrightarrow{0} \in T^{-1}(W)$. Great!

Now for the second rule of subspace, we have to check that if $\overrightarrow{z}, \overrightarrow{u} \in T^{-1}(W)$, then $\overrightarrow{z} + \overrightarrow{u} \in T^{-1}(W)$. Unwinding the definition of what it means to be in $T^{-1}(W)$, this means that $T(\overrightarrow{z}) \in W$ and $T(\overrightarrow{u}) \in W$, and we have to prove that $T(\overrightarrow{z} + \overrightarrow{u}) \in W$. Since T is linear we have $T(\overrightarrow{z} + \overrightarrow{u}) = T(\overrightarrow{z}) + T(\overrightarrow{u})$, and since W is a subspace, $T(\overrightarrow{z}) + T(\overrightarrow{u}) \in W$ by closure under addition. Therefore we have

$$T(\overrightarrow{z} + \overrightarrow{u}) = T(\overrightarrow{z}) + T(\overrightarrow{u}) \in W,$$

and hence $\overrightarrow{z} + \overrightarrow{u} \in T^{-1}(W)$.

Finally, we have to check that $T^{-1}(W)$ is closed under scalar multiplication. So suppose that the devil gives you $\overrightarrow{u} \in T^{-1}(W)$ and some scalar $c \in \mathbb{R}$, and your job is to show $c\overrightarrow{u} \in T^{-1}(W)$. Again, we'll begin by writing down what it means for all these elements to be in $T^{-1}(W)$: we are given that $T(\overrightarrow{u}) \in W$, $c \in \mathbb{R}$ some scalar, and we are supposed to prove that $T(c\overrightarrow{u}) \in W$. Since T is linear we know that $cT(\overrightarrow{u}) = T(c\overrightarrow{u})$, and since W is a subspace we know that $cT(\overrightarrow{u}) \in W$ since W is closed under scalars (remember, we already had that $T(\overrightarrow{u}) \in W$). Therefore

$$T(c\overrightarrow{u}) = cT(\overrightarrow{u}) \in W,$$

and hence $c\overrightarrow{u} \in T^{-1}(W)$. □

Problem. Suppose that $V \subseteq W$, where both are subspace. Show that $\dim(V) \leq \dim(W)$.

Solution. Recall that $\dim(V)$ is the number of elements in a basis of V, and that $\dim(W)$ is the number of elements in a basis for W. So to show the desired inequatlity, let $\overrightarrow{v}_1, \cdots, \overrightarrow{v}_s$ and $\overrightarrow{w}_1, \cdots, \overrightarrow{w}_t$ be bases for V

and W, respectively. (Note that our notation means that $s = \dim(V)$ and $t = \dim(W)$, and we're now aiming to show that $s \leq t$.)

Notice first that both $\vec{v}_1, \cdots, \vec{v}_s$ and $\vec{w}_1, \cdots, \vec{w}_t$ are collections of vectors in the subspace W (we've used the fact that $V \subseteq W$ here). Since $\vec{v}_1, \cdots, \vec{v}_s$ are a basis for V, they form a *linearly independent* collection of vectors in V (and therefore, also a linearly independent collection of vectors in W—think about why this is true). Using the other property of bases, since $\vec{w}_1, \cdots, \vec{w}_t$ is a basis of W, this collection *spans* the space W. Now a proposition in your book says the following: if you have a collection of spanning vectors for a subspace U and another collection of linearly independent vectors inside U, the number of elements in the linearly independent collection is no bigger than the number of elements in the spanning collection (this is Proposition 12.1 rearranged a bit). In our case, this means that $s \leq t$, which is what we wanted to prove. $\square$