

## WHY ARE VECTORS IN DIFFERENT EIGENSPACES INDEPENDENT?

In this note I'll prove the statement I made in the review session tonight. Namely, I'll prove

**Theorem 0.1.** *If  $\vec{v}_1, \dots, \vec{v}_n$  are (non-trivial) eigenvectors for  $A$  with corresponding eigenvalues  $\lambda_1, \dots, \lambda_n$  which are all distinct, then  $\vec{v}_1, \dots, \vec{v}_n$  are linearly independent.*

*Proof.* This proof will require some cleverness, so I doubt something like it will show up on your exam. But it's a fun fact to know and something that you'll likely use, so it's good to see a proof. Let me reemphasize, though, that I think this proof is trickier than something you'd see on the test. So proceed only if you want to be illuminated and are willing to deal with a proof that has a few interesting twists and turns.

Ok, so we want to show that these vectors are linearly independent. Suppose for the sake of contradiction that they are instead dependent. This means that there exists scalars  $c_1, \dots, c_n$ —not all zero—so that

$$c_1 \vec{v}_1 + \dots + c_n \vec{v}_n = \vec{0}.$$

But it doesn't just mean that. There is an old theorem in your book that says that vectors  $\vec{v}_1, \dots, \vec{v}_n$  are linearly dependent if and only if there exists some  $1 \leq i \leq n$  so that  $v_i$  is a linear combination of  $\vec{v}_1, \dots, \vec{v}_{i-1}$ . In other words,  $\vec{v}_1, \dots, \vec{v}_n$  are linearly independent if one of the vectors  $\vec{v}_i$  satisfies

$$\vec{v}_i = c_1 \vec{v}_1 + \dots + c_{i-1} \vec{v}_{i-1}.$$

(This isn't normally a result that we use, so that's why I think this proof is a little tricky).

Ok, so since we've assumed the vectors  $\vec{v}_1, \dots, \vec{v}_n$  are dependent then we can find some  $i$  so that  $\vec{v}_i$  is a linear combination of  $\vec{v}_1, \dots, \vec{v}_{i-1}$ . There might be many  $i$  that satisfy this condition, but let's pick the smallest  $i$  which satisfies this condition. That is,  $i$  is an index so that

$$\vec{v}_i = c_1 \vec{v}_1 + \dots + c_{i-1} \vec{v}_{i-1}$$

holds, but  $\vec{v}_1, \dots, \vec{v}_{i-1}$ . Ok. Remember that.

Now let's take our equation  $\vec{v}_i = c_1 \vec{v}_1 + \dots + c_{i-1} \vec{v}_{i-1}$  and hit it with the  $A$ . We then have the equation

$$A\vec{v}_i = Ac_1 \vec{v}_1 + \dots + Ac_{i-1} \vec{v}_{i-1},$$

which is the same as

$$\lambda_i \vec{v}_i = c_1 \lambda_1 \vec{v}_1 + \dots + c_{i-1} \lambda_{i-1} \vec{v}_{i-1}$$

since the  $\vec{v}_k$  are eigenvectors with corresponding eigenvalues  $\lambda_k$ . Great. But that's not all! In fact, using the first equation we have that the left hand side of this equation is the same as

$$\lambda_i \vec{v}_i = \lambda_i (c_1 \vec{v}_1 + \dots + c_{i-1} \vec{v}_{i-1}) = c_1 \lambda_i \vec{v}_1 + \dots + c_{i-1} \lambda_i \vec{v}_{i-1}.$$

Aha! So we have the following two equations:

$$\lambda_i = c_1 \lambda_i \vec{v}_1 + \dots + c_{i-1} \lambda_i \vec{v}_{i-1}$$

and

$$\lambda_i \vec{v}_i = c_1 \lambda_1 \vec{v}_1 + \dots + c_{i-1} \lambda_{i-1} \vec{v}_{i-1}.$$

Let's go crazy and subtract the two equations. This gives me

$$\vec{0} = c_1 (\lambda_i - \lambda_1) \vec{v}_1 + \dots + c_{i-1} (\lambda_i - \lambda_{i-1}) \vec{v}_{i-1}.$$

But remember that we chose  $i$  really small so that  $\vec{v}_1, \dots, \vec{v}_{i-1}$  are linearly independent! This means that the coefficients

$$c_1(\lambda_i - \lambda_1) = c_2(\lambda_2 - \lambda_i) = \dots = c_{i-1}(\lambda_i - \lambda_{i-1}) = 0.$$

But since the  $\lambda$ 's are all distinct, this means that each  $c_k = 0$ .

But remember that  $c_k$  were the constants so that

$$\vec{v}_i = c_1 \vec{v}_1 + \dots + c_{i-1} \vec{v}_{i-1}.$$

If all these constants are 0 (which we just proved), then this means that  $\vec{v}_i$  has to be the zero vector. But that's impossible since all the  $\vec{v}_k$  were non-trivial by assumption. We've reached a contradiction, which means that our assumption (that we can write some  $\vec{v}_i$  as a linear combination of  $\vec{v}_1, \dots, \vec{v}_{i-1}$ ) is false. Hence we have that all the vectors  $\vec{v}_1, \dots, \vec{v}_n$  are linearly independent. We win!  $\square$

For those of you who think this proof is complicated: it is. So you don't have to worry about it if you don't want to. Again, this was only for people who had a burning desire to know why this fact is true.