In class today we stated the following

Theorem. *Suppose that* $\sum_{n=1}^{\infty} a_n$ *is an alternating series which converges, and such that the hypotheses of the alternating series theorem apply. Then*

$$\left|\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{k} a_n\right| \leq |a_{k+1}|.$$

Notice that the term on the left side of the inequality above represents the error in the kth partial sum approximation for the series $\sum_{n=1}^{\infty} a_n$.

Example. What is the smallest k I can choose in order to know that $\sum_{n=1}^{k}(-1)^n \frac{1}{n^2}$ is within 10^{-6} of $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}$?

Solution. Notice that the series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}$ is alternating, and that

(1) $|a_{n+1}| = \left|(-1)^{n+1}\frac{1}{(n+1)^2}\right| = \frac{1}{(n+1)^2} \leq \frac{1}{n^2} = \left|(-1)^n \frac{1}{n^2}\right| = |a_n|$, since the denominator on the left side of the inequality is larger than the denominator of the right side of the inequality
(2) $\lim_{n\to\infty} \left|(-1)^n \frac{1}{n^2}\right| = \lim_{n\to\infty} \frac{1}{n^2} = 0$.

Hence the hypotheses of the alternating series test apply, and so I can use the error approximation theorem. This tells me that for any given k, we have

$$\left|\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{k} a_n\right| \leq |a_{k+1}|.$$

So if I choose k such that $|a_{k+1}| \leq 10^{-6}$, then this means that the kth partial sum approximation has an error term which is at most 10^{-6} as desired. But

$$|a_{k+1}| = \left|(-1)^{k+1}\frac{1}{(k+1)^2}\right| = \frac{1}{(k+1)^2} < 10^{-6}$$

is equivalent to asking for

$$10^6 < (k+1)^2.$$

Now I can simply solve for k in this expression:

$$10^3 = \sqrt{10^6} < k+1,$$

and so $k > 1000 - 1 = 999$.

So the smallest value of k for which the kth partial sum approximation has an error term smaller than 10^6 is 1000. □

Example. What is the error in using the 10th partial sum approximation for the alternating series $\displaystyle\sum_{n=2}^{\infty}(-1)^n \frac{1}{n\ln(n)}$?

Solution. Notice that the series in question is alternating, and we can verify that the hypotheses of the alternating series test apply:

(1) To show that the (absolute value) of the terms of the series are decreasing, we'll compute a derivative and show it's negative:

$$\frac{d}{dx}\left[\frac{1}{x\ln(x)}\right] = \frac{-1-\ln(x)}{(x\ln(x))^2} < 0$$

(2) $\lim_{n\to\infty}\left|(-1)^n \frac{1}{n\ln(n)}\right| = \lim_{n\to\infty}\frac{1}{n\ln(n)} = 0$ since the denominator approaches ∞ as $n \to \infty$.

Now the alternating series approximation theorem tells us that the error in the 10th partial sum approximation is at most $|a_{11}| = \frac{1}{11\ln(11)} \approx 0.0379$. □