Linear algebra in your daily (digital) life

Andrew Schultz

Wellesley College

March 6, 2012

Alternate title

A TIME-TRAVELER'S GUIDE TO BECOMING A BILLIONAIRE

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Note: the value of this talk has significantly depreciated since 1990

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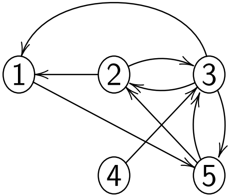
Thanks to Brian White, Jerry Uhl, Bruce Carpenter, Bill Davis, Dan Boyd, ...

The general constraints

Suppose that G is a directed graph



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- V is the vertex set for G
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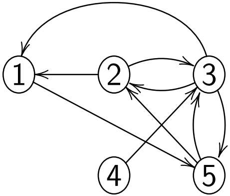

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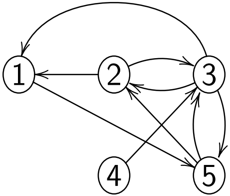

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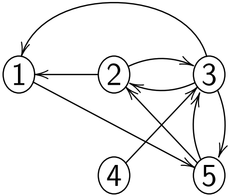

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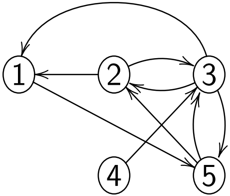

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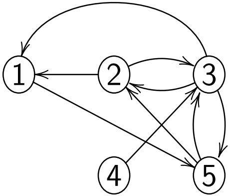

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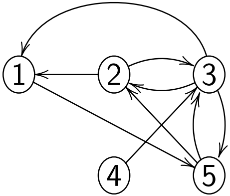

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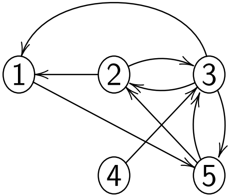

The first approach

$j \longrightarrow i$ is a vote for i from j

Form matrix L so that

$$L_{i,j} = \begin{cases} 1 & \text{, if } j \longrightarrow i \\ 0 & \text{, else} \end{cases}$$

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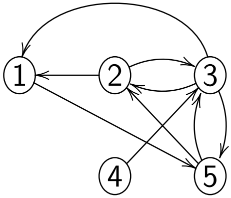

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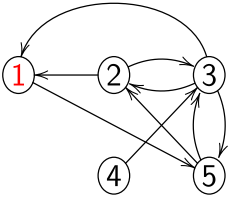

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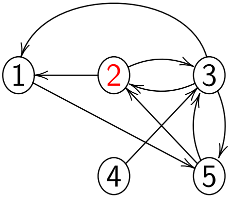

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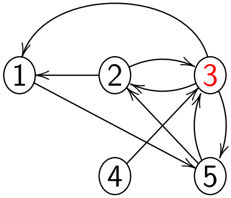

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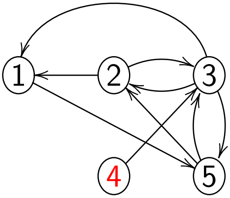

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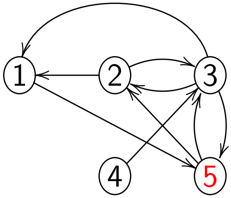

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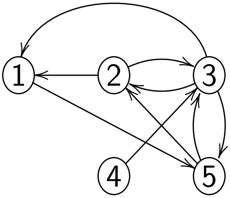

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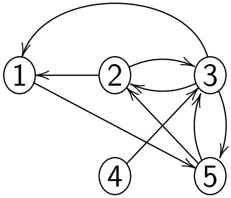

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Page 3 wins (score 3), Pages 1,2,5 tie for second (score 2), Page 4 loses

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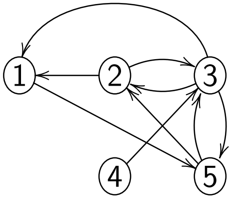

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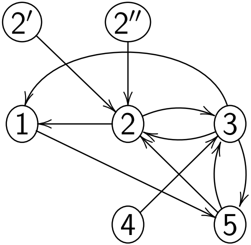

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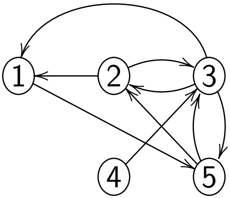
Updating our approach

Each page gets a total of 1 vote:

$$\ell(j) = \sum_i L_{i,j}$$



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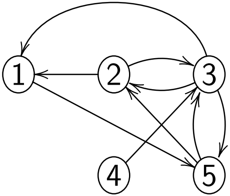

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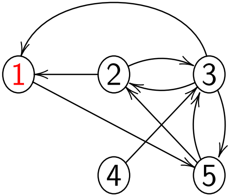

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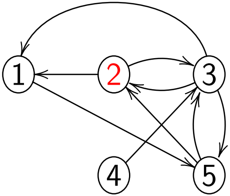

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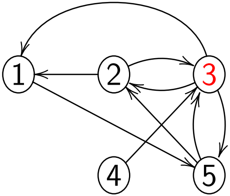

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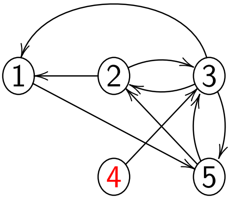

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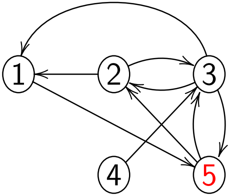

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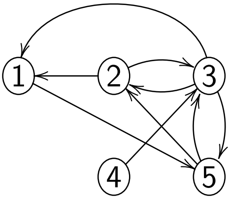

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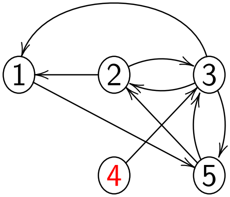

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$$\begin{pmatrix} 0.41 \\ 0.47 \\ 0.52 \\ 0 \\ 0.58 \end{pmatrix}$$ is 1-eigenvector

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Let $PR = dW + (1-d)\begin{pmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \ddots & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}$

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  - If a supercomputer started at the dawn of the universe, then today it would be 0.0000001% done.

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$$\overrightarrow{v} = c_1 \overrightarrow{v_1} + \cdots + c_{10^{13}} \overrightarrow{v_{10^{13}}}.$$

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Hence $\lim_{k\to\infty} PR^k \overrightarrow{v}$ is an eigenvalue with eigenvalue 1.

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If $\sum_{i>s} \sigma_i$ is "insignificant," then we have $A \approx \displaystyle\sum_{i=1}^{s} A_i$

Noise Filtering

Matrix representations of images

You can think of an image as a matrix.

- Each pixel contains a gray value
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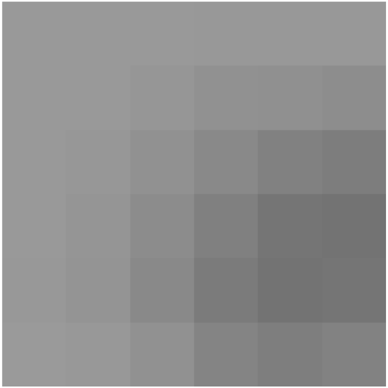

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153	153	153	152	152	152
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153	149	140	128	117	115
152	148	137	123	115	117
154	152	145	132	126	130

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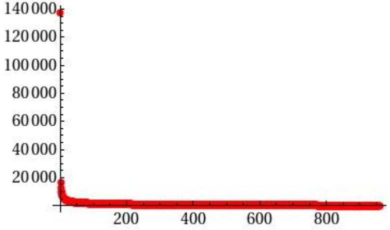
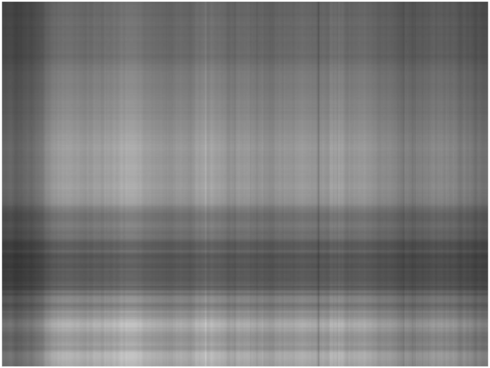


$$\sigma_1 \approx 138,000$$
$$\sigma_2 \approx 17,000$$
$$\sigma_{50} \approx 2,200$$
$$\sigma_{200} \approx 900$$

Some approximations

Let's see what our truncated matrix "looks like"

- s=1
- Compression: 0.18%



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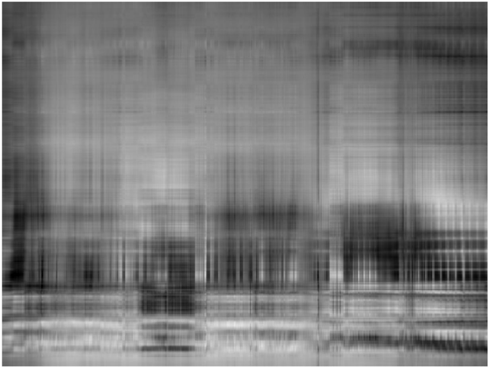
- s=5
- Compression: 0.9%



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- s=10
- Compression: 1.8%

Some approximations

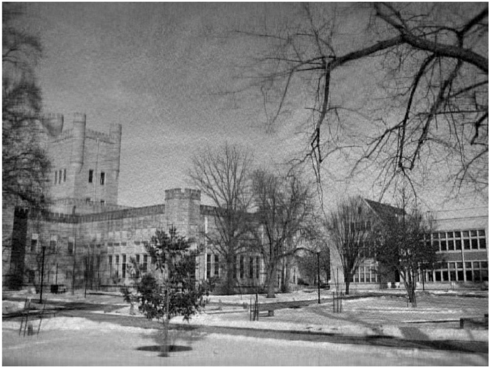
Let's see what our truncated matrix "looks like"

- s=25
- Compression: 4.5%

Some approximations

Let's see what our truncated matrix "looks like"

- s=100
- Compression: 18%



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- takes advantage of properties of images

The "usual" way of thinking about a matrix

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- Deleting an a_{ij} completely wipes out pixel info.

Rewriting the matrix

What if we chose a different basis for $n \times n$ matrices?

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- deleting $c_{i,j}$ has gradual (though global) effect

A potential basis

We'll choose an orthonormal basis of $\mathbb{R}^n$ from Fourier series

$$\vec{f}_i = \alpha_i \left\{ \cos\left[\frac{\pi}{n}\left(\frac{2j+1}{2}\right) i\right]\right\}_{j=0}^{n-1}$$

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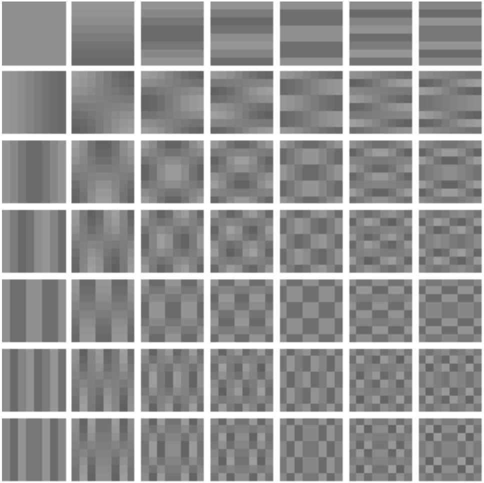
$$B(i,j) = \left(\begin{array}{c} \vec{f}_0 \\ \hline \vdots \\ \hline \vec{f}_{n-1} \end{array}\right) E(i,j) \left(\begin{array}{c|c|c} \vec{f}_0 & \cdots & \vec{f}_{n-1} \end{array}\right)$$

Seeing the new basis ($n = 8$)



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Then decompression is

- De-quantize
- Change back to standard coordinates
- Reassemble the 8×8 blocks

Working through an example

Extract an 8×8 block

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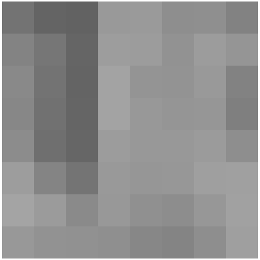

$$\longrightarrow \begin{pmatrix} 115 & 100 & 98 & 153 & 154 & 142 & 143 & 130 \\ 131 & 118 & 101 & 157 & 156 & 146 & 156 & 149 \\ 137 & 115 & 100 & 163 & 148 & 147 & 153 & 130 \\ 135 & 113 & 101 & 163 & 152 & 149 & 150 & 127 \\ 140 & 111 & 102 & 156 & 152 & 152 & 155 & 142 \\ 157 & 132 & 116 & 153 & 150 & 151 & 159 & 160 \\ 164 & 155 & 138 & 152 & 144 & 141 & 151 & 161 \\ 152 & 146 & 145 & 143 & 135 & 132 & 142 & 159 \end{pmatrix}$$

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We're down to 16 pieces of information!

Reconstituting our image

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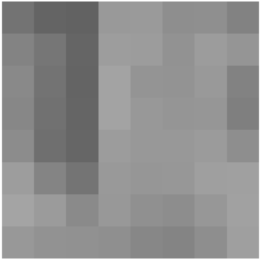
Compressed

$$\begin{pmatrix} 114 & 103 & 101 & 144 & 149 & 136 & 154 & 135 \\ 125 & 111 & 106 & 150 & 156 & 142 & 57 & 134 \\ 133 & 115 & 107 & 151 & 160 & 146 & 157 & 130 \\ 134 & 114 & 104 & 147 & 158 & 146 & 157 & 129 \\ 139 & 118 & 106 & 148 & 156 & 146 & 163 & 139 \\ 151 & 132 & 118 & 153 & 155 & 145 & 169 & 153 \\ 162 & 144 & 129 & 155 & 147 & 135 & 165 & 158 \\ 166 & 150 & 132 & 152 & 137 & 122 & 157 & 154 \end{pmatrix}$$

- Average difference $= 5.73$
- Std Dev $= 4.22$

Seeing is believing

Original



Compressed

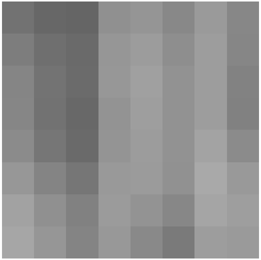


Image Processing

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Note: Here I won't split the image into 8×8 blocks – I want all the information about the image simultaneously

Image Processing

Smooth Part: $B(i, j)$ components for small j, small i

Image Processing

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Image Processing

Horizontal Edges: $B(i,j)$ components for small j, large i

Image Processing

Horizontal Edges: $B(i,j)$ components for small j, large i

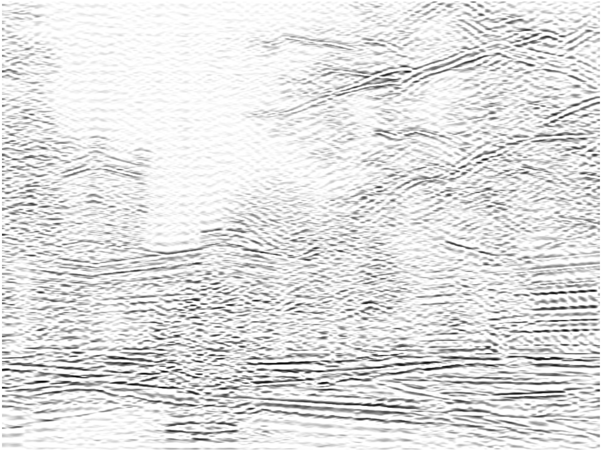


Image Processing

Vertical Edges: $B(i, j)$ components for small i, large j

Image Processing

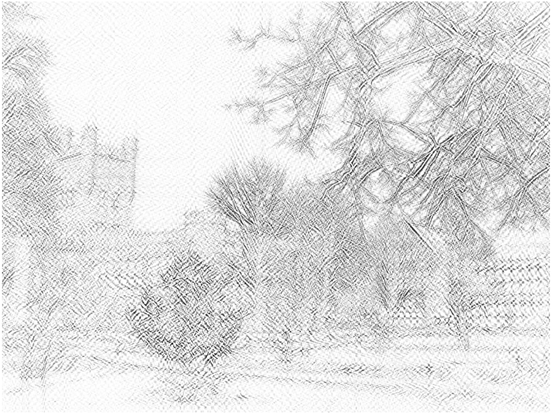
Vertical Edges: $B(i,j)$ components for small i, large j

Image Processing

Scattered Edges: $B(i,j)$ components for large i, large j

Image Processing

Scattered Edges: $B(i,j)$ components for large i, large j



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Similar ideas are used to compress music into mp3's

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- Filter out "unnecessary" data using psychoacoustics

Psychoacoustics

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- Temporal Masking - Some weak sounds aren't heard if played right after (or right before!) a louder sound
- Hass effect - If the same tone hits one ear just before another, then your brain perceives it as coming only from the first direction

Thanks!

Thank you!