MATH 51: MATLAB HOMEWORK 1

1. Working with MATLAB

Matrices play a central role in linear algebra, though even the seasoned master encounters situations in which explicitly computing with matrices becomes burdensome. For this reason several computer programs have been designed to perform otherwise tedious computations. One of the most powerful of these is MATLAB (short for *matrix laboratory*), and in this section we introduce some of the basic operations that MATLAB performs.

1.1. **Starting MATLAB.**

When you start MATLAB, the MATLAB desktop appears, containing tools for managing files, variables, and associated applications. The most important of these windows is the *Command Window*, located on the right-half of the screen. This is where we'll be entering our commands into MATLAB, and it's also where the results will display.

When describing operations in MATLAB, we will use the convention

```
>> input
output
```

to mean `input` was entered into MATLAB, and upon executing the input (by hitting enter) MATLAB produces `output`.

1.2. **Matrices in MATLAB.**

Let's start by defining a matrix. If you've just started up the program, the cursor should be blinking on the *command line*, which is indicated by a double arrow, >>. To enter a matrix in MATLAB, you only have to follow a few basic conventions:

- Separate the elements of a row with blanks or commas.
- Use a semicolon, **;** , to indicate the end of each row.
- Surround the entire list of elements with square brackets, [ ].

Example. The matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is defined in MATLAB using:

```
>> A=[1,2;3,4]
A =
     1     2
     3     4
```

Performing operations on matrices, such as multiplication, is very simple. Let's start by defining a second matrix, $B = \begin{pmatrix} 3 & 2 \\ 5 & 6 \end{pmatrix}$, by entering the command:

```
>> B=[3,2;5,6]
B =
     3     2
     5     6
```

To multiply the matrices A and B, simply enter the command:

```
>> A*B
ans =
    13    14
    29    30
```

Thus, we've computed that $AB = \begin{pmatrix} 13 & 14 \\ 29 & 30 \end{pmatrix}$.

One can also use MATLAB to compute high powers of a matrix. The matrix A^{10}, for instance, is computed by

```
>> A^10
ans =
     4783807      6972050
    10458075     15241882
```

Another useful operation is finding the *eigenvalues* and *eigenvectors* of a matrix. To find the eigenvalues and eigenvectors of our matrix A, enter the following command:

```
>> [V,E]=eig(A)

V =
   -0.8246   -0.4160
    0.5658   -0.9094

E =
   -0.3723         0
         0    5.3723
```

In this output, the columns of V correspond to eigenvectors of A, with eigenvalue given on the diagonal of E. For instance, in our example, A has eigenvector $\begin{pmatrix} -0.8246 \\ 0.5658 \end{pmatrix}$ with eigenvalue -0.3723, and eigenvector $\begin{pmatrix} -0.4160 \\ -0.9094 \end{pmatrix}$ with eigenvalue 5.3723.

Caution: With its default settings, MATLAB outputs the eigenvalues to four decimal places (even though the values are computed internally with much greater precision). In cases where a matrix has both very small and very large eigenvalues, this can result in "false 0" eigenvalues.

For example, suppose we try to compute the eigenvalues of $(AB)^6$ in our above example.

```
>> [V,E]=eig((A*B)^6)
V =
   -0.7232   -0.4187
    0.6906   -0.9081

E =
  1.0e+009 *

    0.0000         0
         0    6.6538
```

At first glance, it appears that the eigenvalues of $(AB)^6$ are 0.0000 and 6.6538×10^9. To check this, we can ask MATLAB to explicitly output the $(1,1)$-coordinate of E:

```
>> E(1,1)
ans =
    0.0025
```

So, the actual eigenvalues of $(AB)^6$ are 0.0025 and 6.6538×10^9. Be sure to be on the lookout for such "false 0" eigenvalues in your own computations.

1.3. **Exercises.** In the following exercises,

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

Use MATLAB to perform the following calculations, and record the MATLAB output below.

(1) Compute $(AB)^{20}$

(2) Find the eigenvalues for $(BA)^6$, and for each eigenvalue a corresponding eigenvector.

2. An Introduction to Markov Processes

For the remainder of this problem set we will be investigating Markov processes. This section describes a specific Markov Process, shows how one uses matrices to answer questions about the system, and finally gives a Markov process for you to work out in detail.

2.1. **A Problem.** In Happytown, the local newspaper *The Happy Gazette* has determined that a citizen who purchases a copy of their paper on one day has a 70% chance of buying the following day's edition. They have also determined that a person who does not purchase a copy of *The Happy Gazette* one day has a 20% chance of purchasing a copy the next. Records show that of the 1000 citizens of Happytown, exactly 750 purchased a copy of the paper on Day 0. To determine the appropriate amount of papers to press each day, the owner of the *Gazette*, Happy Golucky, is interested the following types of questions:

(1) If a person purchased a paper today, how likely is he to purchase a paper on Day 2? Day 3? Day n?
(2) What sales figures can the *Gazette* expect on Day 2? Day 3? Day n?
(3) Will the sales figures fluctuate a great deal from day to day, or are they likely to become stable eventually?

2.2. **The Solution.** Happy Accounting has been brought in to answer these questions, and their newest accountant, Susan Smiles, says she has a clever way to determine the *Gazette*'s future sales figures. She begins by dividing the population of Happytown into two groups: group 1 are all those citizens who read the paper on a given day, and group 2 are all those citizens who do not read the paper on a given day. She keeps track of the size of each of these groups on Day n using a vector

$$\mathbf{v_n} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \mathbb{R}^{\mathbf{2}},$$

where g_1 is the size of group 1 on Day n and g_2 is the size of group 2 on Day n.

She now constructs a 2x2 matrix $M = (m_{ij})$, where the entry m_{ij} gives the probability that a citizen in group j on one day will belong to group i the next. For instance, m_{21} gives the probability that a person who purchased a paper one day will not buy a paper the next. Using the sales figures, she sees that $m_{11} = 0.7$, and likewise $m_{12} = 0.2$. She is also able to deduce $m_{21} = 0.3$ because a person who buys a paper one day and fails to purchase a paper the next is not among the 70% of citizens who purchase a paper two days in a row. Similarly, $m_{22} = 0.8$, and so the matrix M is

$$\begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}.$$

The power of this setup, she explains, is that

$$M\mathbf{v_n} = \mathbf{v_{n+1}}.$$

Indeed, suppose g_1 people buy a paper on Day n and g_2 people do not buy a paper on Day n. Since 70% the people who purchase a paper on Day n return as customers on Day $n+1$, and 20% of people who didn't purchase a paper on Day n do purchase a paper on Day $n+1$, we see that there are $0.7 * g_1 + 0.2 * g_2$ people who purchase a paper on Day $n+1$. Likewise, there are $0.3 * g_1 + 0.8 * g_2$ people who do not purchase a paper on Day $n+1$. Since

$$M\mathbf{v_n} = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 0.7 * g_1 + 0.2 * g_2 \\ 0.3 * g_1 + 0.8 * g_2 \end{pmatrix} = \mathbf{v_{n+1}},$$

we see that Susan's observation is correct.

Using her idea and the initial sales data for Day 0, Susan can compute $\mathbf{v_n}$ as $M^n\mathbf{v_0}$, since

$$\begin{aligned}M^n\mathbf{v_0} &= (M^{n-1}M)\mathbf{v_0} = M^{n-1}(M\mathbf{v_0}) = M^{n-1}\mathbf{v_1}\\ &= (M^{n-2}M)\mathbf{v_1} = M^{n-2}(M\mathbf{v_1}) = \mathbf{M^{n-2}v_2}\\ &\quad\vdots\\ &= (M^{n-(n-1)}M)\mathbf{v_{n-2}} = M^{n-(n-1)}(M\mathbf{v_{n-2}}) = M\mathbf{v_{n-1}} = \mathbf{v_n}.\end{aligned}$$

Happy Golucky asks Susan for projected sales figues for Days 5, 10, and 100.

Solution. Susan computes these figures using MATLAB, beginning with the computation:

```
>> M=[0.7,0.2;0.3,0.8];v=[750;250];
>> M^5*v
ans =
  410.9375
  589.0625
```

This tells Susan that the paper can expect to sell approximately 411 papers on Day 5. Similar computations show that on Days 10 and 100, the *Gazette* can expect to sell 400 papers.

While Susan is happy with her clever idea, she becomes especially excited when she notices that i, j entry of the matrix M^2 gives the probability that a person in group j will belong to group i in 2 days time; the i, j entry of M^3 gives the probability that a person in group j will belong to group i in 3 days time; etc.

Happy Golucky returns to Susan and asks her to compute

- the probability that a person who purchases a paper one day will not purchase a paper 5 days later, and
- the probability that a person who does not purchase a paper one day will not purchase a paper 2 weeks later.

Solution. Using her observation from above, Susan knows the answer to the first question is given by the entry in the 2nd row, 1st column of the matrix M^5. She computes this using MATLAB:

```
>> M^5
ans =
    0.4188    0.3875
    0.5813    0.6125
```

Hence the probability that a person who buys a paper one day will not purchase a paper the next is approximately 58.1%.

Similarly, the probability that a person who does not buy a paper one day will not purchase a paper two weeks later is given by the entry in the 2nd column, 2nd row of the matrix M^{14}. Using MATLAB, she sees the probability is approximately 60%.

Susan has one last question to answer before her work for the *Gazette* is complete, and that is to determine if and when production numbers for the paper will stablize. After she considers the situation, Susan realizes that production will be stable when $\mathbf{v_n} = \mathbf{v_{n+1}} = \mathbf{v_{n+2}} = \cdots$ for some n. Since $\mathbf{v_{n+1}} = M\mathbf{v_n}$, however, stable production is equivalent to the seemingly less powerful

$$M\mathbf{v_n} = \mathbf{v_n}.$$

In the language of linear algebra, stable production is equivalent to the vector $\mathbf{v_n}$ being an eigenvector for M with eigenvalue 1, or at least that $\mathbf{v_n}$ is very nearly an eigenvector for M with eigenvalue 1.

Kyle the Joyful, the mechanic for *The Happy Gazette*, is worried that fluctuating production numbers will do damage to the paper's printing press. He asks Susan when he can expect paper production to stabilize and how many papers he will need to produce once stable production is reached.

Solution. Using the logic above, Susan knows production will stabilize as the vectors v_n approach an eigenvector of M with eigenvalue 1. Susan uses MATLAB to find the eigenvectors and eigenvalues of the matrix M.

```
>> [V,E]=eig(M)
V =
   -0.7071   -0.5547
    0.7071   -0.8321
E =
    0.5000         0
         0    1.0000
```

She sees that the eigenspace corresponding to eigenvalue 1 is spanned by the vector $\begin{pmatrix} -0.5547 \\ -0.8321 \end{pmatrix}$. So, Susan is looking for a vector $\mathbf{v} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ in the span of $\begin{pmatrix} -0.5547 \\ -0.8321 \end{pmatrix}$, which also satisfies $g_1 + g_2 = 1000$ (since $g_1 + g_2$ represents the total number of people). To find this eigenvector $\mathbf{v}$, she writes $\mathbf{v} = \lambda \begin{pmatrix} -0.5547 \\ -0.8321 \end{pmatrix}$, and attempts to solve for λ. First, she notes that

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \mathbf{v} = \lambda \begin{pmatrix} -0.5547 \\ -0.8321 \end{pmatrix} \implies g_1 = -0.5547\lambda, \quad g_2 = -0.8321\lambda.$$

Substituting this into the condition $g_1 + g_2 = 1000$, she finds that

$$1000 = g_1 + g_2 = -0.5547\lambda - 0.8321\lambda = -1.3868\lambda \implies \lambda \approx -721.08.$$

Thus, she finds that $g_1 = -0.5547\lambda \approx 399.98$ and $g_2 = -0.8321\lambda \approx 600.01$. Rounding to the nearest integers (since g_1 and g_2 represent numbers of people), Susan concludes that the desired eigenvector is roughly $\begin{pmatrix} 400 \\ 600 \end{pmatrix}$.

So, in the long run, the number of people purchasing the paper will stabilize to 400. Note that this agrees with her earlier calculations, where Susan found that 400 people would purchase the *Gazette* on Day 10, and the same number on Day 100.

2.3. **Exercises.** Each Friday, the 543 students in Math 51 are given the choice of attending one of three calculus parties: Party 1 (which features free food and drinks), Party 2 (where a $100 prize is given each week), and Party 3 (where students work extra problems for fun). Previous experience shows

- Of the students who attend Party 1 one week, 50% remain in Party 1 the next week and 25% move to Party 2.
- Of the students who attend Party 2 one week, 70% remain in Party 2 the next week and 25% move to Party 3.
- Of the students who attend Party 3 one week, 10% move to Party 2 the next week and 10% move to Party 1.

Attendence at exactly one party each week is mandatory in the course. To begin the semester, Professor Brubaker decides all students must attend Party 1 on Friday of Week 0.

Answer the following questions using the Markov Process techniques described above.

(1) Determine the number of students who will attend Party 3 on Week 5.

(2) What is the probability that a student who attends Party 3 on a given week will attend Party 1 two weeks later? That a student who attends Party 3 on a given week will attend Party 3 again four weeks later?

(3) Does the number of students who attend each party stabilize by Friday of Week 3? Explain why or why not.