MATH 52: MATLAB HOMEWORK 2

1. Complex Numbers

The prevalence of the complex numbers throughout the scientific world today belies their long and rocky history. Much like the negative numbers, complex numbers were originally viewed with mistrust and skepticism. In fact, the term "imaginary number" was a derogatory term coined by René Descartes, who found such numbers unsettling. Just as with the negative numbers, however, history has shown that not only are the complex numbers immensely useful, they can also have physical meaning, and hence are no less "real" than the real numbers. Perhaps more importantly, they have given deep insight into a broad range of problems and become an indispensable tool both for mathematicians and scientists in general.

Complex numbers first gained prominence in the 16^{th} century, when Italian mathematicians Niccolo Tartaglia and Gerolamo Cardano discovered formulas for the roots of general cubic and quartic polynomials. Surpisingly, their formulas required the use of square roots of negative numbers, even when using the formulas to find real roots of polynomials. You've likely encountered similar situations in your own studies when trying to find roots of quadratic polynomials. Recall the following familiar theorem:

Quadratic Formula. Suppose $f(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$ and $a \neq 0$. If $b^2 - 4ac < 0$, then f has no real roots. If $b^2 - 4ac \geq 0$, then the roots of f are precisely

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quantity $b^2 - 4ac$ is sometimes called the *discriminant* of f, and written $\operatorname{disc}(f)$.

Example.

Suppose $f(x) = x^2 + 1$. Then $\operatorname{disc}(f) = -4 < 0$, so f has no real roots.

This example is not purely artificial. In fact, you might have come across this scenario when searching for eigenvalues of linear transformations.

Example.

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation which rotates the plane 90^o counterclockwise about the origin. The matrix for T with respect to the standard basis is

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of this matrix is $p(\lambda) = \lambda^2 + 1$. By the previous example, it follows that A has no real eigenvalues.

If we restrict ourselves to the real numbers, we see that not every polynomial has a root (and hence not every linear transformation has an eigenvalue). This seems quite unfortunate, especially since the quadratic formula seems to suggest what the roots (or eigenvalues) "should" be. To remedy this situation, we will enlarge our collection of numbers to include some of these "imaginary" roots. We'll start by simply adding $\sqrt{-1}$.

Definition. Let i represent a "number" which satisfies $i^2 = -1$. We call i the *imaginary unit*.

We'll now add i to our collection of numbers.

Definition. We define the *complex numbers* to be the set

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}.$$

We define addition and multiplication of complex numbers using the relation $i^2 = -1$. That is, given complex numbers $a_1 + ib_1, a_2 + ib_2$, we define

$$(a_1 + ib_1) + (a_2 + ib_2) := (a_1 + a_2) + i(b_1 + b_2)$$

and

$$(a_1 + ib_1)(a_2 + ib_2) := (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1).$$

Examples.

(1) Suppose $f(x) = x^2 + 4$. We claim the roots of f are $x = \pm 2i$. Indeed, observe that

$$f(\pm 2i) = (\pm 2i)^2 + 4 = 4i^2 + 4 = 4(-1) + 4 = 0.$$

Note that this agrees with the roots predicted by the quadratic formula: $x = \frac{\pm\sqrt{-16}}{2} = \frac{\pm\sqrt{16}\sqrt{-1}}{2} = \pm 2i$.

(2) Suppose $f(x) = x^2+2x+3$. Using the quadratic formula, the predicted roots of f are $x = \frac{-2\pm\sqrt{-8}}{2} = -1 \pm i\sqrt{2}$. As a check, we compute

$$f(-1 + i\sqrt{2}) = (-1 + i\sqrt{2})^2 + 2(-1 + i\sqrt{2}) + 3 = \Big((1 - 2) + i(-2\sqrt{2})\Big) - 2 + i2\sqrt{2} + 3 = 0.$$

MatLab can natively work with complex numbers. For example, the above computation is entered as follows:

```
>> (-1+sqrt(2)*i)^2+2*(-1+sqrt(2)*i)+3
ans =

 -4.4409e-016
```

Notice that MatLab has introduced a small rounding error, outputting a result of $-4.4409\text{x}10^{-16} \approx 0$.

Exercise 1. Use MatLab to compute $(1 - 2i)^5$.

You might be worried we will need to keep adding new "numbers" to our collection as we attempt to find roots of polynomials of higher degrees. The following theorem, known as the Fundamental Theorem of Algebra, ensures we won't need to extend our collection any further.

Theorem. *Let $p(x) = a_nx^n + \cdots + a_1x + a_0$ be a polynomial, with $a_i \in \mathbb{C}$ and $a_n \neq 0$. Then*

$$p(x) = a_n(x - r_1)\cdots(x - r_n)$$

for some $r_1, \ldots, r_n \in \mathbb{C}$.

In other words, when working with the complex numbers, every polynomial of degree n has n (not necessarily distinct) roots. This theorem may seem more amazing in light of the fact there do not exist formulas for the roots of general polynomials of degree five or more; that is, we can't prove the theorem by simply writing down the roots of a general polynomial and checking they're all in our set $\mathbb{C}$.

Example.

The roots of $x^2 + 1$ are $x = \pm i$, which we can express as

$$x^2 + 1 = (x - i)(x + i).$$

In general, finding the roots of a polynomial is quite difficult. Fortunately, MatLab can do this for us. First we must define our polynomial. MatLab stores polynomials as row vectors, with components given by the coefficients of the monomials. For example, to store the polynomial $p(x) = x^5 - 2x + 3$, we enter

```
>> p=[1 0 0 0 -2 3];
```

To find the roots of p we use the `roots` command:

```
>> r=roots(p)
r =

  -1.4236
  -0.2467 + 1.3208i
  -0.2467 - 1.3208i
   0.9585 + 0.4984i
   0.9585 - 0.4984i
```

As predicted by the previous theorem, the degree five polynomial p has five complex roots.

Exercise 2. Use MatLab to find the roots of the polynomial $p(x) = x^4 - 2x^3 + 1$.

Since we'll be working with the complex numbers, it will be useful to have a few additional definitions.

Definition. Suppose $z \in \mathbb{C}$ is given by $z = a + ib$, with $a, b \in \mathbb{R}$. We make the following definitions:

(1) The real number a is called the *real part* of z, and is denoted $\text{Re}(z)$. A complex number z with $\text{Re}(z) = 0$ is called *purely imaginary*.
(2) The real number b is called the *imaginary part* of z, and is denoted $\text{Im}(z)$. A complex number z with $\text{Im}(z) = 0$ is called *purely real*.
(3) The *complex conjugate* of z, denoted by $\bar{z}$, is given by

$$\bar{z} = a - ib = \text{Re}(z) - i\text{Im}(z).$$

Example. MatLab has a built-in function for conjugation, called `conj`. For example:

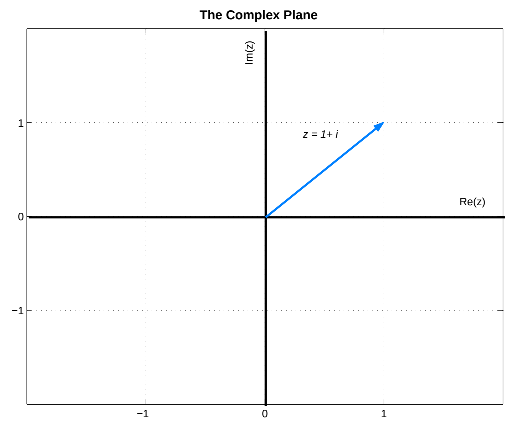
```
>> conj(-1+2*i)
ans =

  -1.0000 - 2.0000i
```

Exercise 3. Prove that $\text{Re}(z) = \frac{1}{2}(z + \bar{z})$ and $\text{Im}(z) = \frac{1}{2i}(z - \bar{z})$.

By the above exercise, any equation involving the real and imaginary parts of z can be written in terms of z and $\bar{z}$. This observation will be useful to remember when we discuss complex functions in Section 2.

It is often helpful to picture the complex numbers as lying in the plane, with the real part along the horizontal axis and the imaginary part along the vertical axis.



Unsurprisingly, this plane is called the *complex plane*. It was first described by Caspar Wessel in 1799, although it is also sometimes referred to as the *Argand plane* and credited to Jean-Robert Argand. Use of the complex plane was later popularized by Carl Gauss, and it was only after this geometric interpretation was introduced that the complex numbers became widely accepted. The complex plane can be used to give polar coordinates for z, with which one can deduce many beautiful relations. Unfortunately, this would take us beyond the scope of the current assignment.

2. Complex Functions

Now that we've extended our collection of numbers to the field of complex numbers, we can begin studying complex-valued functions. The attempt to extend the theory of calculus to such functions is the branch of mathematics known as *complex analysis*. In the complex world, both differentiation and integration gain new depth and significance. As a sacrifice, however, we must severely limit the class of functions we can analyze, restricting to so-called *analytic* (or *holomorphic*) functions. For a rough definition, suppose $f(x, y)$ is any complex-valued function of two real variables. Let $z = x + iy$. We call z the *complex variable*. By Exercise 3, we can use the identities $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$ to express $f(x, y)$ as a function of z and $\bar{z}$. Morally, the function f is *analytic* if it does not depend on $\bar{z}$. The analytic functions are exactly the complex functions for which a derivative exists.

Example. MatLab can work directly with complex functions. For example, to find the roots of $f(z) = z^2 + 2iz + 3$, we compute:

```
>> p=[1 2*i 3];
>> r=roots(p)
r =

        0 - 3.0000i
   0.0000 + 1.0000i
```

Thus, the roots are $z = -3i$ and $z = i$.

Exercise 4. Find the roots of $f(z) = z^4 - iz^3 + 2i$.

It is a fact that all complex polynomials are analytic. We can use polynomials to construct more general complex functions. For instance, consider a quotient of two complex polynomials, say $f(z) = \frac{p(z)}{q(z)}$. For simplicity, assume p and q have distinct roots. It is then a fact that f is analytic away from the roots of q, which we call the *poles* of f. More generally, we have the following definition:

Definition. Suppose f is a function which is analytic in a neighborhood of a point a, except perhaps at a itself. If $\lim_{z\to a} f(z) = \infty$, then the point a is said to be a *pole* of f.

By the following fact, the nature of a general pole is exactly the same as in the case of a rational function:

Fact. If f has a pole at a, then there exists a positive integer n and analytic function g not vanishing at a such that

$$f(z) = \frac{g(z)}{(z-a)^n}.$$

The integer n is called the *order* of the pole. Poles of order one are called *simple* poles. A function which is analytic away from a discrete set of poles is a called *meromorphic*. A function which is analytic everywhere is sometimes called *entire*.

Example.

By the previous example, we know the function $f(z) = \frac{1}{z^2+2iz+3}$ is a meromorphic function with simple poles at $z = -3i$ and $z = i$.

Exercise 5. What are the poles of $f(z) = \frac{z^2-2z+1}{z^5+4z^3+z-2}$?

Associated to every pole of a meromorphic function f is a special number called the *residue* of f at that pole.

Definition. Suppose f is a meromorphic function with a pole at a. The *residue* of f at a is the unique complex number R such that

$$f(z) - \frac{R}{z-a}$$

is the derivative of an analytic function in a neighborhood of a, excluding a itself. We'll usually denote this number by $\text{Res}_{z=a} f(z)$.

This definition may seem rather arbitrary, but in Section 3 we'll see that residues are intimately linked with integration.

Example.

Fortunately, MatLab can directly calculate residues of rational functions using the `residue` command. For example, to compute the residues of $f(z) = \frac{1}{z^2+2iz+3}$, we enter:

Example (continued).

```
>> p=[1];
>> q=[1 2*i 3];
>> [r, a, k]=residue(p,q)
r =

  -0.0000 + 0.2500i
   0.0000 - 0.2500i


a =

        0 - 3.0000i
   0.0000 + 1.0000i


k =

     []
```

MatLab has outputted the residues `r` at the corresponding poles `p`. In this case, we see that $\text{Res}_{z=-3i} f(z) = 0.25i$ and $\text{Res}_{z=i} f(z) = -0.25i$.

Exercise 6. Find the poles and corresponding residues of the function $f(z) = \frac{z+1}{z^3-2z+2}$.

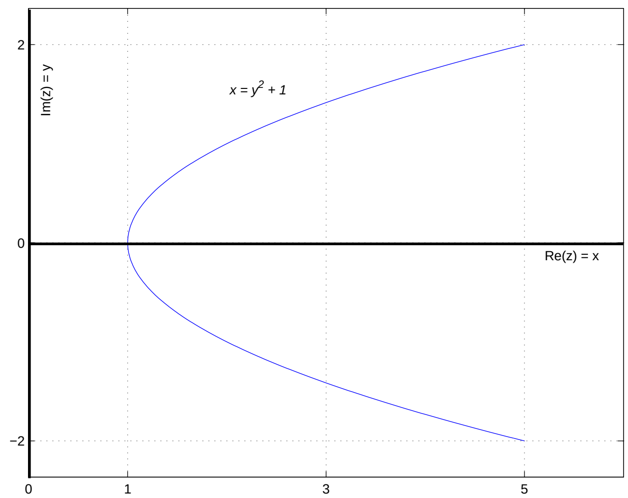
3. Contour Integrals

A complex function can be integrated along a path in the complex plane much in the same way a real function of two real variables can be integrated along a path in $\mathbb{R}^2$. In contrast to the real case, however, these complex integrals (called *contour integrals*) are intimately linked with the poles of the integrand. Before stating the precise relationship, let's compute some contour integrals and make a few initial discoveries. We use the following definition.

Definition. Suppose C is a path in the complex plane, and $f(z)$ is a complex function. Given a parametric representation $z(t)$ of C, with $t_0 \leq t \leq t_1$, we define

$$\int_C f(z)\, dz := \int_{t_0}^{t_1} f(z(t)) \frac{dz}{dt}(t)\, dt.$$

Example. Let's calculate $\int_C z^2 - 2z + 1\ dz$, where C is the contour given below:



The curve C is given by the equation $x = y^2 + 1$. We must first find a parameterization of C. Suppose we introduce the parameter t, and let $y = t$ for $-2 \le t \le 2$. Since $x = y^2 + 1$, we then have $x = t^2 + 1$. We therefore have a parameterization of the curve C, given by

$$z(t) = (t^2 + 1) + it$$

for $-2 \le t \le 2$.

To store this contour in MatLab we enter:

```
>> syms t real;
>> y = t;
>> x = t^2+1;
>> z = simple(x + i*y)
z =

t^2+1+i*t
```

Now that we have a parameterized path, MatLab can calculate the contour integral of any complex function along this path (so long as that function doesn't have a pole on the path). First we store the function and integrand:

```
>> f = z^2-2*z+1;
>> Integrand = f*diff(z,t);
```

We calculate the contour integral using the `int` command, and convert the result to a numerical answer using the `double` command:

```
>> F = int(Integrand, 't', -2, 2);
>> F = double(F)
F =

        0 +58.6667i
```

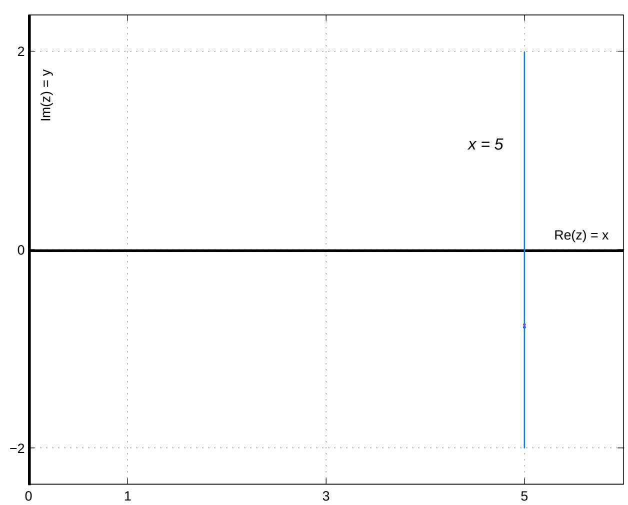
Example (continued).

Thus, $\int_C z^2 - 2z + 1 \; dz \approx 58.6667i$. Of course, we could also calculate this integral explicitly:

$$\begin{aligned}\int_C z^2 - 2z + 1 \; dz &= \int_{-2}^{2} (z(t)^2 - 2z(t) + 1)z'(t) \; dt \\ &= \int_{-2}^{2} \left((t^2+1+it)^2 - 2(t^2+1+it)+1\right)(2t+i) \; dt \\ &= \int_{-2}^{2} 2t^5 + 5it^4 - 4t^3 - it^2 \; dt \\ &= 58\frac{2}{3}i.\end{aligned}$$

As we will soon see, it is a theorem of complex analysis that the contour integral of an entire function depends only on the endpoints of the path, and not on the path taken between them.

Example. To see this in action, let's calculate $\int_{C'} z^2 - 2z + 1 \; dz$ using the contour below, given by the equation $x = 5$, shown below:



We repeat the same process used in the previous example:

```
>> syms t real;
>> y=t;
>> x=5;
>> z=simple(x+i*y);
>> f=z^2-2*z+1;
>> Integrand=f*diff(z,t);
>> F=int(Integrand, 't', -2, 2);
>> F=double(F)
F =

        0 +58.6667i
```

Example (continued).

Thus, $\int_{C'} z^2 - 2z + 1 \ dz \approx 58.6667i$, and so it indeed appears that $\int_C z^2 - 2z + 1 \ dz = \int_{C'} z^2 - 2z + 1 \ dz$. Indeed, we can again manually compute

$$\begin{aligned}\int_{C'} z^2 - 2z + 1 \ dz &= \int_{-2}^{2} \left(z(t)^2 - 2z(t) + 1\right) z'(t) \ dt \\ &= \int_{-2}^{2} (16i - 8t - it^2)(i) \ dt \\ &= 58\frac{2}{3}i \\ &= \int_C z^2 - 2z + 1 \ dz.\end{aligned}$$

It turns out that a contour integral of a meromorphic function f around a closed loop C only depends on the poles of f contained inside C. More precisely, we have the following theorem:

Theorem. *Suppose f is a meromorphic function and C is a simple closed loop (e.g. a circle). If $p_1, \ldots, p_k$ are the poles of f lying in the interior of C, then*

$$\int_C f(z) \ dz = 2\pi i \sum_{i=1}^{k} \mathrm{Res}_{z=p_i} f(z).$$

This remarkable theorem has several corollaries.

Corollary 1. *If f is an entire function, then $\int_C f(z) \ dz = 0$ for any closed loop C.*

Proof. For a simple closed curve C, the statement follows immediately from the theorem. For an arbitrary closed loop C, it is a consequence of a slightly more general version of the theorem. □

Corollary 2. *If f is an entire function, then $\int_C f(z) \ dz$ depends only on the endpoints of C.*

Proof. Suppose C_1 and C_2 are two different paths with the same endpoints. Let C be the loop given by $C_1 - C_2$. Then

$$\int_C f(z) \ dz = \int_{C_1} f(z) \ dz - \int_{C_2} f(z) \ dz.$$

On the other hand, by Corollary 1 we have

$$\int_C f(z) \ dz = 0,$$

since f doesn't have any poles. The result follows.

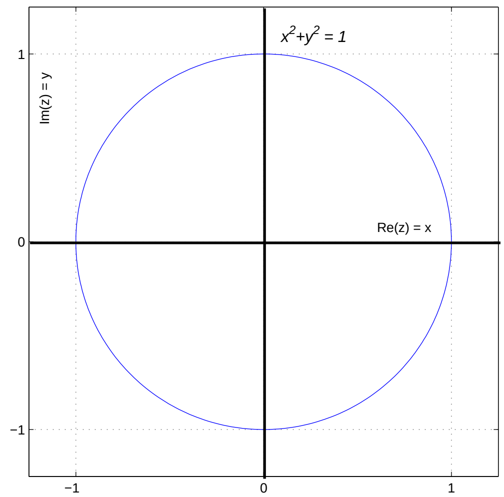
□

Exercise 7. Let C be the contour given by $x^2 + y^2 = 1$, shown below. Observe that C can be parameterized by $x = \cos(t)$, $y = \sin(t)$, for $0 \le t \le 2\pi$.

For each of the functions $f(z)$ given below, use MatLab to:

a) Compute $\int_C f(z) \ dz$ using MatLab;
b) Find the poles of f contained inside C;

c) Compute the residue of f at each of these poles; and
d) Compute $2\pi i \sum_p \mathrm{Res}_{z=p} f(z)$, and compare the result with your answer from part (a).



i) $f(z) = z^3 + 2z^2 + 1$

ii) $f(z) = z^5 - z$

iii) $f(z) = \frac{1}{z^2+\frac{1}{4}}$

iv) $f(z) = \frac{1}{(z+2i)(z+5)}\ dz$

v) $\frac{1}{(z-\frac{i}{9})^3}$