COURSE NOTES - 01/07/05

1. Notation

There were a few notational things that I forgot to cover in Wednesday's class.

During the term we will be refering to sets quite a bit, and so it is important we have some standard notation. Sets will be written inside curly braces $\{\}$, and are usually described in two parts. The first gives the realm from which the elements in our set will be chosen, and the second gives the distinguishing characteristic which elements in our set satisfy. These two parts are separated by a colon. Here are a few examples.

- $\{x \in \mathbb{R} : x \neq 2, 3\}$ is "the set of all real numbers x such that x is neither 2 nor 3."
- $\{x \in \mathbb{R} : x \text{ is an integer}\}$ is "the set of all real numbers x such that x is also an integer." We could also denote this set by $\{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$.
- $\{x \in \mathbb{R} : x \geq 0\}$ is "the set of all real numbers x such that $x \geq 0$." These might also be called the set of non-negative numbers.

We've slipped in another symbol that we'll be seeing a lot of this term: $\mathbb{R}$. This stands for the set of all real numbers.

One of the most basic types of sets we'll be studying this term is an interval. There are a few different flavors of intervals:

(1) The open interval. $(a, b) := \{x \in \mathbb{R} : a < x < b\}$. In words, this is the set of all numbers between a and b, non-inclusive.
(2) The closed interval. $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$. In words, this is the set of all numbers between a and b, inclusive.

You can also mix and match to make 'half-open/half-closed' intervals, such as

$$(a, b] := \{x \in \mathbb{R} : a < x \leq b\}.$$

2. Function addition and multiplication

There are lots of ways to put old functions together to make new functions. Two of the more basic approaches are function addition and function multiplication:

Function addition.	$(f + g)(x) := f(x) + g(x)$
Function multiplication.	$(fg)(x) := f(x)g(x)$

Example. Let $f(x) = 2^x$ and $g(x) = x^2+1$. Then $(f+g)(x) = 2^x+x^2+1$ and $(fg)(x) = 2^x \cdot (x^2+1)$.

Function addition and multiplication have the nice property that they are commutative operations. This means that the order in which I add two functions is not important, and similarly the order in which I multiply two functions is not important. In mathematical notation, commutativity means $(f+g)(x) = (g+f)(x)$ and $(fg)(x) = (gf)(x)$.

We also saw that for any function $f(x)$ there is another, magical function we call $(-f)(x)$ that has the property that

$$(f + (-f))(x) = 0.$$

You might not be too surprised to see that $(-f)(x)$ is defined to be $-f(x)$. The identity above just says the believable statement that $f(x) - f(x) = 0$, independent of x. $(-f)(x)$ is known as the *additive inverse* of $f(x)$.

Example. Again, let $f(x) = 2^x$ and $g(x) = x^2 + 1$. Then $(-f)(x) = -(2^x)$ and $(-g)(x) = -(x^2 + 1) = -x^2 - 1$.

A similar concept exists for functions $f(x)$ which are nowhere 0 (i.e., functions for which $f(x) \neq 0$ for any x). For such a function there is another, magical function we call $\left(\frac{1}{f}\right)(x)$ that has the property that

$$\left(f \cdot \frac{1}{f}\right)(x) = 1.$$

This magical function is defined by $\left(\frac{1}{f}\right)(x) := \frac{1}{f(x)}$, so that the identify above just says that $\frac{f(x)}{f(x)} = 1$, independent of x (notice we have used $f(x) \neq 0$ in order to divide by it). The function $\left(\frac{1}{f}\right)(x)$ is known to its friends as the *multiplicative inverse* of $f(x)$.

Example. We again work with our friends $f(x) = 2^x$ and $g(x) = x^2+1$. Then $\left(\frac{1}{f}\right)(x) = \frac{1}{2^x} = 2^{-x}$ and $\left(\frac{1}{g}\right)(x) = \frac{1}{x^2+1}$. Notice that it is important that $f(x)$ and $g(x)$ are both nowhere 0.

Remark. We will also be running in to functions of the form $\frac{1}{f(x)}$ for functions which might have solutions to $f(x) = 0$. In this case, it is understood that the domain of the function is $\{x \in \text{Domain}(f) : f(x) \neq 0\}$.

3. Function composition

Function addition and multiplication were fine and dandy, but you might feel like we don't have any exciting ways of putting together two functions to make a new function. You would be dead wrong, because function composition will blow....your....mind. We write the composition of two functions $f(x)$ and $g(x)$ as $(f \circ g)(x)$, and read 'f composed with g of x,' or—for the especially trendy—'f of g of x.' We define this composition as

$$(f \circ g)(x) = f(g(x)).$$

In words, we evaluate $f \circ g$ at x by first evaluating g at x, and then plugging the resultant value into f.

Example. Let $f(x) = 2^x$ and $g(x) = x^2 + 1$. Then

$$(f \circ g)(2) = f(g(2)) = f(2^2 + 1) = f(5) = 32.$$

On the other hand

$$(g \circ f)(2) = g(f(2)) = g(2^2) = g(4) = 4^2 + 1 = 17.$$

In the above example, we see that function composition is not commutative! In other words, the order in which I compose two functions *does* matter; if I compose in two different orders, I could very well get two different functions.

Example. With $f(x) = 2^x$ and $g(x) = x^2 + 1$,

$$(f \circ g)(x) = f(g(x)) = f(x^2 + 1) = 2^{x^2+1},$$

while

$$(g \circ f)(x) = g(f(x)) = g(2^x) = (2^x)^2 + 1 = 2^{2x} + 1.$$

4. Inverses

Just as the additive inverse and multiplicative inverse gave us a way to 'undo' the effects of adding or multiplying by a particular function, there is an object which attempts to undo the effect of composing by a particular function. This mysterious object is called the inverse of the function.

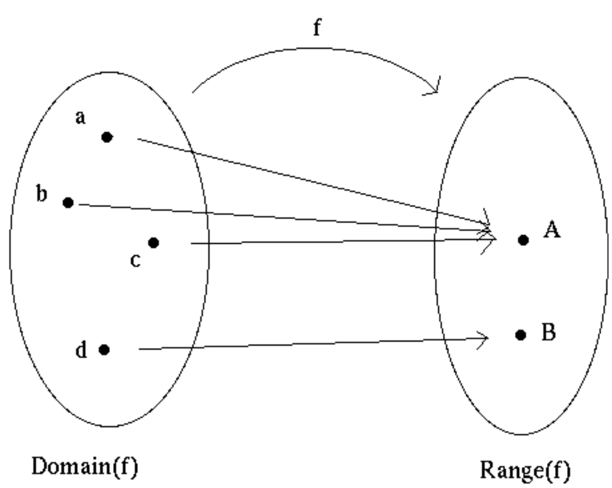
Definition. For a function $f(x)$, the inverse of f, denoted $f^{-1}(x)$, is defined by

$$f^{-1}(x) = \{a \in \text{Domain}(f) : f(a) = x\}.$$

In words, the inverse of f evaluated at x returns the set of all elements in the domain of f which map to x under f.

Warning! Don't be tempted to think that $f^{-1}(x) = \dfrac{1}{f(x)}$. This second quantity would be written $[f(x)]^{-1}$.

Example. Consider the function pictured in the following diagram.



Here, we have $f^{-1}(A) = \{a, b, c\}$ and $f^{-1}(B) = \{d\}$.

This example illustrates one of the important points to remember when working with the inverse of a function: the inverse might not itself be a function. In this last example, we saw that $f^{-1}(A)$ returns *three* values instead of one.

Is the inverse a function? One can test if the inverse of a function is a function a few different ways:

- Graphically. If you are given the graph of a function $f(x)$, you can test to see if $f^{-1}(x)$ is a function by administering the horizontal line test. This is much like the vertical line test, although (you guessed it!) one uses a horizontal line instead. If any horizontal line cuts the graph of the function in more than one place, the inverse is not a function. If all horizontal lines pass through the graph in exactly one place, then the inverse is a function.
- Algebraically. One can test to see if the inverse of a function is a function by determining if the function $f(x)$ is *one-to-one*. A one-to-one function is a function which gives distinct outputs for any two distinct inputs: if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. If you can show that $f(x)$ satisfies this property, then you an be assured $f^{-1}(x)$ is a function.

One trick to make the inverse of a function into a function is to restrict the domain of the function $f(x)$. For instance, the function $f(x) = x^2$ with domain $\mathbb{R}$ has an inverse which is not a function: for any $a \in \mathbb{R}$, we have $f^{-1}(a) = \pm\sqrt{a}$. However, if I restrict the domain of my function to all non-negative numbers, then the inverse is a function: it returns only the positive square root since only the positive square root is in the domain of the function.

Is this an inverse? I often take an afternoon walk, a favorite function $f(x)$ my only companion. Frequently I am stopped by other pedestrians, each accompanied by his or her favorite function $g(x)$, and they always ask me "Andy, is my function the inverse of your function?" Fortunately, I know a test to determine the answer to this question. I know that our functions are inverses of each other if and only if their composition is the function x:

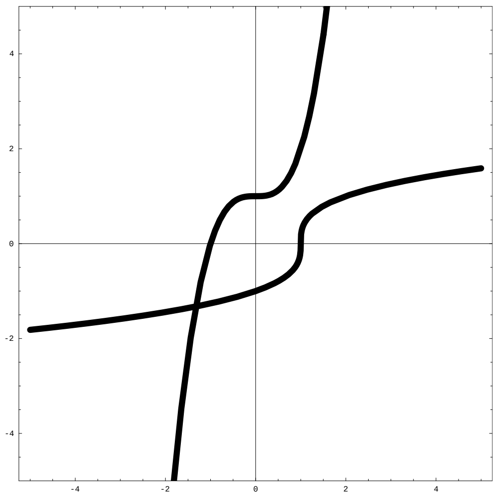
$$g = f^{-1} \text{ if and only if } (f \circ g)(x) = (g \circ f)(x) = x.$$

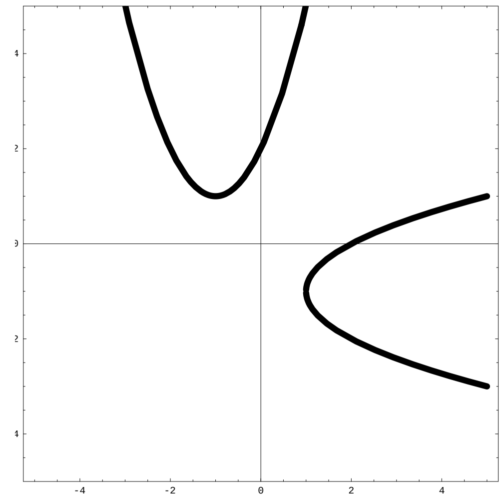
Example. Suppose $f(x) = 2x + 1$ and $g(x) = \dfrac{x-1}{2}$. We will shwo that $g = f^{-1}$ by computing $f \circ g$ and $g \circ f$.

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x-1}{2}\right) = 2\left(\frac{x-1}{2}\right) + 1 = x - 1 + 1 = x$$

$$(g \circ f)(x) = g(f(x)) = g\,(2x+1) = \frac{(2x+1)-1}{2} = \frac{2x}{2} = x$$

Inverses and graphs Given the graph of a function $f(x)$, one can graph its inverse $f^{-1}(x)$ in a fairly straight-forward way. One first sketches the line $y = x$, and then reflects the graph of $f(x)$ across this line. I did an example with chalk in the class (and I'll do a few more monday), but here are a few more.





5. Logarithms

One of the most imporant inverses we'll come across is the logarithm. We define the logarithm to be the inverse of the map $f(x) = e^x$: $\log x$ is that value b so that $e^b = x$. You might have also seen $\log x$ by the name $\ln x$, and you're more than welcome to use this notation if you like. If $f(x) = a^x$ for some other a which is not e, the inverse is written $\log_a x$.

The logarithm is, indeed, a function, and—as with all inverses—its domain is the range of the function from which it is defined. In this case, since the range of $f(x) = e^x$ is the set of all positive numbers $\{x \in \mathbb{R} : x > 0\}$, the domain of the logarithm map is $\{x \in \mathbb{R} : x > 0\}$.

Two very important identities, and the domains on which they hold, come out of the exponential and logarithm map. Hold them close to your heart!

Identity	Domain
$e^{\log x} = x$	$(0, \infty)$
$\log(e^x) = x$	$(-\infty, \infty)$

Some properties of the logarithm map, and the corresponding statement about exponentials, which you should feel comfortable with:

$$\log(xy) = \log x + \log y \quad \leftrightarrow \quad e^a e^b = e^{a+b}$$

$$\log\left(\frac{x}{y}\right) = \log x - \log y \quad \leftrightarrow \quad \frac{e^a}{e^b} = e^{a-b}$$

$$\log(1) = 0 \quad \leftrightarrow \quad e^0 = 1$$