COURSE NOTES - 01/10/05

1. Announcements

1.1. **Course webpage.** The course webpage is finally up and running. You can find it at

$$http://math.stanford.edu/\sim aschultz/math19$$

The course webpage has PDF files of all handouts given in class. It will also be the place where course homework is posted. After an assignment has been turned in, you can go to the course webpage to find solutions. Finally, I hope to have my course notes posted on the course website as well; no promises, but my personal goal is to have class notes for a Monday class posted by Monday night, a Wednesday class posted by Wednesday night, and a Friday class posted before the end of the weekend.

1.2. **HW1.** The first homework has been posted, and is due at the *start* of class on Friday. Late homework will not be accepted, so make sure you're in class on time. Also as a reminder: when writing up solutions for homework, be sure to work on your own. Put away any notes you may have taken from office hours or discussions with friends and recreate the solutions by yourself.

1.3. **Office Hours.** Finally, office hours for the term have been set and are posted on the course webpage. And, as always, remember that if you'd like to meet me sometime outside of office hours, you're more than welcome! Your chances of running into me outside of office hours are greatly improved if you send me an email to let me know you're coming (preferably the day before), but if you're willing to risk my not being around you can come without notice!

2. Recap

Last Friday we dove deeper into the mystical ocean of functions. Memorable scenery included

- ways of putting together functions to make a new function, including function addition and multiplication.
- a new fangled way of putting together two functions, called *function composition*. The composition of functions $f(x)$ and $g(x)$ is written $(f \circ g)(x)$, and is defined by $(f \circ g)(x) = f(g(x))$.
- the inverse of a function, including questions like
  When will the inverse of a function be a function itself?
  What does the graph of an inverse look like?
  How can I test if I have an inverse of a given function?
- the logarithm function

3. Tying up loose ends

We are nearly ready to climb onto the dry land of Calculus, our home for the rest of the quarter. Before we do this, we need to take care of a few more facts about functions.

3.1. Properties of the logarithm.

3.1. Properties of the logarithm. There are 3 key facts you should keep in mind when working with a logarithm function. I will state these facts as properties of the natural logarithm, but analogous statements hold for the logarithm of any base. I am also listing the corresponding fact of exponential functions which make these properties work.

$$\begin{array}{lcl} \log(xy) = \log x + \log y & \leftrightarrow & e^a e^b = e^{a+b} \\ \log\left(\dfrac{x}{y}\right) = \log x - \log y & \leftrightarrow & \dfrac{e^a}{e^b} = e^{a-b} \\ \log(1) = 0 & \leftrightarrow & e^0 = 1 \end{array}$$

3.2. More graphs of inverses. In class we saw two more examples of how one can sketch the graph of $f^{-1}(x)$ given the graph of $f(x)$. They used the techniqe we developed on Friday:

a. Sketch the line $y = x$.
b. Reflect the graph of $f(x)$ across this line.

For some pictures similar to those shown in class, take a look at Friday's coursenotes.

3.3. Computing inverses. It often happens that a friend of yours will ask you to compute the inverse of his or her favorite function. Happily, there is a pretty nice trick that lets you do just that. The steps are as follows

(1) Write your friend's function in the form $y = f(x)$
(2) In this expresssion, replace y by x and x by y
(3) Solve this new expression for y

We saw a few examples of this in class. As well as I can remember, Jeremy and Matt conspired to give the following problem.

Problem 1. Let $f(x) = 13x + 8$. Compute $f^{-1}(x)$.

Solution. We'll follow the steps outlined above. First, we write $y = f(x) = 13x + 8$. Second, we exchange y and x; this gives us a new expression

$$x = 13y + 8.$$

Finally, we solve for y in this last expression. Some basic algebra gives

$$y = g(x) = \frac{x-8}{13}.$$

The function $g(x)$ should be the inverse of $f(x)$. Since we want to make sure we have the correct solution, we'll verify:

$$(g \circ f)(x) = g(f(x)) = g(13x+8) = \frac{(13x+8)-8}{13} = \frac{13x}{13} = x$$

and

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x-8}{13}\right) = 13\left(\frac{x-8}{13}\right) + 8 = x - 8 + 8 = x.$$

$\square$

Problem 2. Let $f(x) = 7x^2 + 4x - 12$. Compute $f^{-1}(x)$.

Solution. Again, we follow the steps outlined above. After exchanging x and y, we're left to solve for y in the equation

$$x = 7y^2 + 4y - 12.$$

Solving for y looks like it might cause a massive headache. Before we get too discouraged, though, we notice this is a quadratic expression in the variable y, so we can use the quadratic formula to find solutions. Since the equation I'm trying to solve is

$$0 = 7y^2 + 4y - 12 - x,$$

my solutions are

$$y = \frac{-4 \pm \sqrt{16 - 4(7)(-12 - x)}}{14} = \frac{-4 \pm \sqrt{352 + 28x}}{14}.$$

Notice that in this case the inverse is *not a function*! $\square$

3.4. **Even and odd functions.** One final pair of creatures is left to be investigated in this land of functions: even functions and odd functions.

Definition. A function $f(x)$ is said to be an *even function* if $f(-x) = f(x)$.

Definition. A function $f(x)$ is said to be an *odd function* if $f(-x) = -f(x)$.

Examples. The function $g(x) = x^4 + x^2 + 3$ is an even function, since

$$g(-x) = (-x)^4 + (-x)^2 + 3 = x^4 + x^2 + 3 = g(x).$$

The function $\cos x$ is also an even function.

The function $f(x) = x^3 + x$ is an odd function, since

$$f(-x) = (-x)^3 + (-x) = -(x^3 + x) = -f(x).$$

The functions $\sin x$ and $\tan x$ are also odd functions.

We did an example in class which combined all our knowledge of odd functions and inverse functions.

Problem 3. Suppose that $f(x)$ is an *odd* function. Complete the following table

x	-3	-2	-1	0	1	2	3
$f(x)$	1				3		
$f^{-1}(x)$						-2	

Solution. We'll fill in some entries on the top row first.

- Since $f(x)$ is an odd function and $f(-3) = 1$, we have $f(3) = f(-(-3)) = -f(-3) = -1$.

- Similarly, since $f(1) = 3$, we have $f(-1) = -f(1) = -3$.
- Since f is odd, we also have $f(0) = -f(-0) = -f(0)$, and we conclude $f(0) = 0$.

Now looks like

x	-3	-2	-1	0	1	2	3
$f(x)$	1		-3	0	3		-1
$f^{-1}(x)$						-2	

Now $f^{-1}(-2) = 2$ means '$\{2\}$ is the set of inputs with output -2.' This means we have $f(2) = -2$. We can now use the fact that f is odd to conclude $f(-2) = -f(2) = -(-2) = 2$. Our table now looks like

x	-3	-2	-1	0	1	2	3
$f(x)$	1	2	-3	0	3	-2	-1
$f^{-1}(x)$						-2	

So we still have to compute $f^{-1}(x)$ for $x \in \{-3, -2, -1, 0, 1, 3\}$.

- $f^{-1}(-3)$ asks the questions 'What inputs have output -3.' We see from our previous computations that $f(-1) = -3$, and that no other values of x satisfy $f(x) = -3$, so $f^{-1}(-3) = -1$.
- $f^{-1}(-2)$ asks the questions 'What inputs have output -2.' We see from the top row that $f(2) = -2$, and that no other values of x satisfy $f(x) = -2$, so $f^{-1}(-2) = 2$.
- $f^{-1}(-1)$ asks the questions 'What inputs have output -1.' We see from the top row that $f(3) = -1$, and that no other values of x satisfy $f(x) = -1$, so $f^{-1}(-1) = 3$.
- $f^{-1}(0)$ asks the questions 'What inputs have output 0.' We see from the top row that $f(0) = 0$, and that no other values of x satisfy $f(x) = 0$, so $f^{-1}(0) = 0$.
- $f^{-1}(1)$ asks the questions 'What inputs have output 1.' We see from the top row that $f(-3) = 1$, and that no other values of x satisfy $f(x) = 1$, so $f^{-1}(1) = -3$.
- $f^{-1}(2)$ asks the questions 'What inputs have output 2.' We see from the top row that $f(-2) = 2$, and that no other values of x satisfy $f(x) = 2$, so $f^{-1}(2) = -2$.
- $f^{-1}(3)$ asks the questions 'What inputs have output 3.' We see from the top row that $f(1) = 3$, and that no other values of x satisfy $f(x) = 3$, so $f^{-1}(3) = 1$.

So our table is completed as

x	-3	-2	-1	0	1	2	3
$f(x)$	1	2	-3	0	3	-2	-1
$f^{-1}(x)$	-1	2	3	0	-3	-2	1

$\square$

4. A first look at calculus

With our mastery of functions and algebra, we are now ready to explore Calculus! Amazingly, the ‘big ideas’ of calculus can almost all be seen from behind a steering wheel.

When driving, we’re used to knowing how fast we’re going at any one moment. Because of this, people are accustomed to thinking of speed as a quantity that can be computed instantaneously. In practice, however, how would a person accomplish this?

Suppose that an object moves a given distance in an given interval of time. The average velocity of this object over the given interval of time is computed as

$$\text{Average Velocity} = \frac{\text{distance traveled}}{\text{time it took to travel distance}}.$$

The idea for computing instantaneous velocity is a pretty simple one. If we’re interested in computing the instantaneous velocity of an object at a time t_0, we simply measure how far the object travels in a very small interval of time, say $[t_0, t_1]$. We can then compute the velocity of our object over this small time interval. The intuition is that the smaller the time interval (that is, the closer t_1 is to t_0), the better the approximation of the instantaneous speed will be. When translated into the language of functions, this amounts to finding the ‘instantaneous slope’ of a function at a given point. It connects to an old question from analytic geometry.

Old Question. Given the graph of a function $f(x)$ and a point $P = (x, f(x))$ on the graph of $f(x)$, compute the equation of the line passing through the point P and tangent to the graph of x.

We will discuss on Wednesday an intuitive attack on this problem, which will lead us to investigate limits.