COURSE NOTES - 01/14/05

1. Announcements

- No class on Monday, January 17. Enjoy the holiday!

2. Recap

In the previous lecture we revisited the tangent problem and used it as a pathway into investigating limits. Specifically, we

- stated the tangent problem;
- defined an intuitive approach to solving the tangent problem via secant lines;
- saw we needed to understand the quantity
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$
as x_2 approached x_1 (i.e., as the secant line through $(x_1, f(x_1))$ and $(x_2, f(x_2))$ approached the tangent line through $(x_1, f(x_1))$);
- gave an intuitive definition for limits, including directional limits;
- computed the limits of certain functions by inspecting graphs;
- saw how to guess a limit based on a chart of information;
- learned to be skeptical of limits which were 'determined' from a chart.

3. Limit laws

The big results of the day give us a way of computing the limit of sums, differences, products, and quotients of functions if we know the limits of the summands or factors. In particular, we saw 5 limits laws. In each case, *it is implicit that each of the stated limits exist!* If you want to evalue the limit of a function using these rules and one of your component functions does not have a limit at the point in question, you cannot apply these rules directly! The rules are:

(1) (Sums) $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$. The content of this rule is 'the limit of a sum is the sum of the limits.'
(2) (Differences) $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$. The content of this rule is 'the limit of a difference is the difference of the limits.'
(3) (Scalers) For c a (fixed) real number, $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$. One could say that 'scalars pass through the limit.'
(4) (Products) $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$. The content of this rule is 'the limit of a product is the product of the limits.'

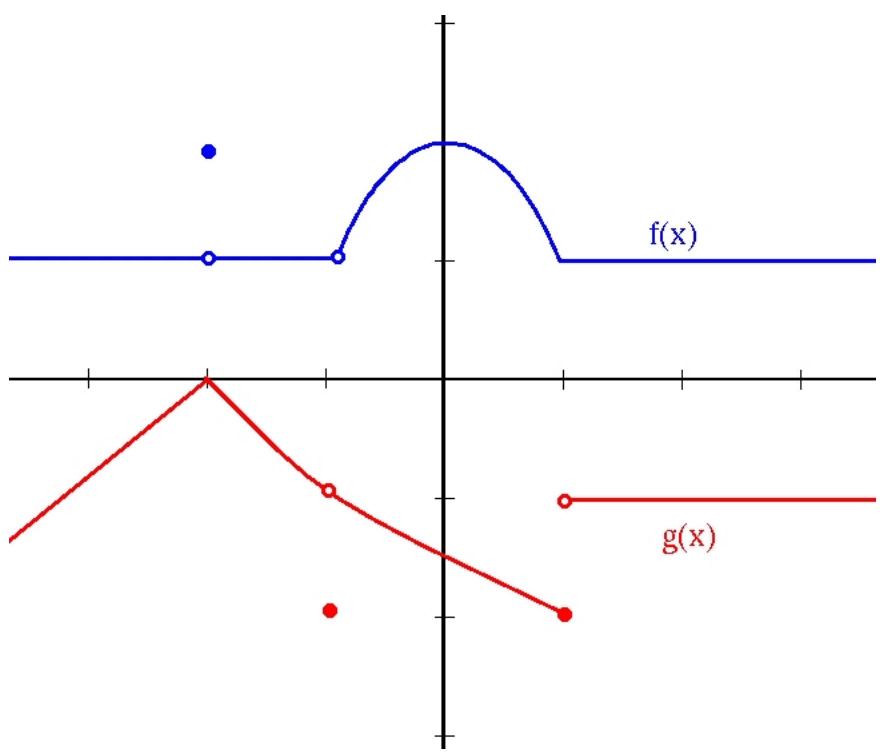
(5) (Quotients) Suppose $\lim_{x\to a} g(x) \neq 0$. Then $\lim_{x\to a}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$. The content of this rule is 'the limit of a quotient is the quotient of the limits.'

Although we won't rewrite them here, the corresponding results also hold for directional limits. So, for instance, we have

$$\lim_{x\to a^+}[f(x)+g(x)] = \lim_{x\to a^+} f(x) + \lim_{x\to a^+} g(x),$$

provided each of these directional limits exists.

Example 1. Consider the functions $f(x)$ and $g(x)$ shown below.

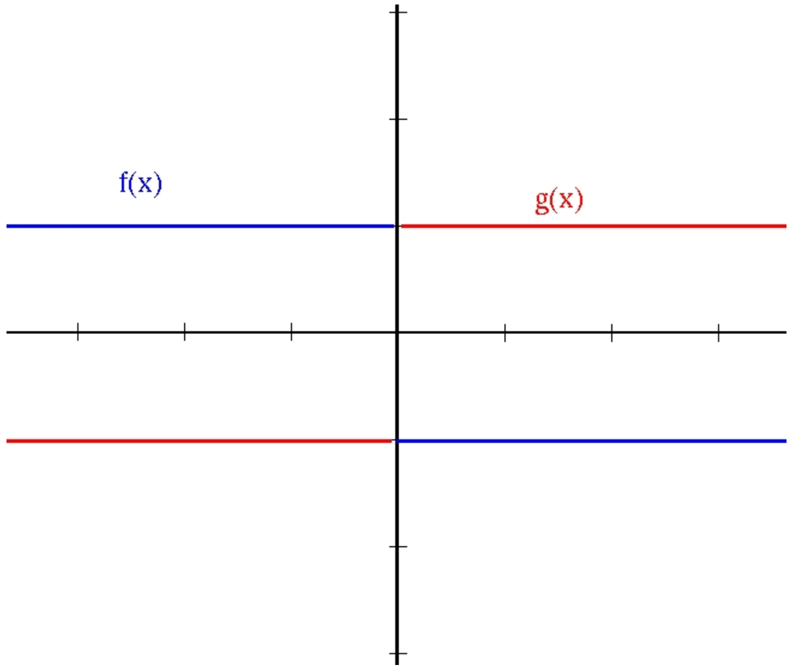


Then one can evaluate $\lim_{x\to -2} f(x) = 1$, $\lim_{x\to -2} g(x) = 0$. Hence we have

$$\lim_{x\to -2}[3f(x)+2g(x)] = \lim_{x\to -2}[3f(x)] + \lim_{x\to -2}[2g(x)] = 3\lim_{x\to -2} f(x) + 2\lim_{x\to -2} g(x) = 3(1)+2(0) = 3.$$

In a similar way one might evaluate $\lim_{x\to -2} f(x)g(x)$, but note that one cannot use the rules above to compute $\lim_{x\to -2} \dfrac{f(x)}{g(x)}$ since $\lim_{x\to -2} g(x) = 0$.

Example 2. Consider the functions $f(x)$ and $g(x)$ shown below.



Notice to evaluate $\lim_{x\to 0} f(x)+g(x)$, one cannot use the limits laws above since both $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist. However, one can calculate $\lim_{x\to 0^-} f(x) = 1$ and $\lim_{x\to 0^-} g(x) = -1$; using the limit laws for directional limits, we have

$$\lim_{x\to 0^-} f(x)+g(x) = \lim_{x\to 0^-} f(x) + \lim_{x\to 0^-} g(x) = 0.$$

Similarly, one can see $\lim_{x\to 0^+} f(x) = -1$ and $\lim_{x\to 0^+} g(x) = 1$; using the directional limit laws it follows $\lim_{x\to 0^+} f(x)+g(x) = \lim_{x\to 0^+} f(x) + \lim_{x\to 0^+} g(x) = 0$. Since $\lim_{x\to 0^-} f(x)+g(x) = 0 = \lim_{x\to 0^+} f(x)+g(x)$, we have

$$\lim_{x\to 0} f(x)+g(x) = 0.$$

How interesting: each summand has no limit as $x \to 0$, though the sum does! You'll have a homework problem like this for products.

4. Evaluating limits of some familiar functions

The limit laws we have discussed so far are fantastic because they will allow us to compute the limits of a whole bevy of functions without having to draw graphs. Two limits which we will take for granted are

(1) $\lim_{x\to a} c = c$, where c is some fixed real number, and
(2) $\lim_{x\to a} x = a$.

Each of these limits can be seen easily by drawing the graph of either the constant function $f(x) = c$ or the identity function $f(x) = x$.

Given these facts, however, we can use our skills at manipulating limits to evaluate a whole class of limits.

Example 3. Evaluate $\lim_{x\to a} x^3 + x^2 + 4$.

$$\begin{aligned}
\lim_{x\to a} x^3 + x^2 + 4 &= \lim_{x\to a} x^3 + \lim_{x\to a} x^2 + \lim_{x\to a} 4 && \text{(limit law 1)}\\
&= \lim_{x\to a} x \cdot \lim_{x\to a} x \cdot \lim_{x\to a} + \lim_{x\to a} x \cdot \lim_{x\to a} + \lim_{x\to a} 4 && \text{(limit law 4)}\\
&= a \cdot a \cdot a + a \cdot a + 4 && \text{(1 \& 2 above)}\\
&= a^3 + a^2 + 4.
\end{aligned}$$

Notice that for this function $p(x) = x^3 + x^2 + 4$ we have $\lim_{x\to a} p(x) = p(a)$. We have seen many times that, generally speaking, the quantity $\lim_{x\to a} f(x)$ is not related to $f(a)$. For our example, however, we are in the fortunate situation where $\lim_{x\to a} p(x) = p(a)$. You can see that a similar argument will give the following exciting result.

Fact 1. If $f(x)$ is a polynomial, then $\lim_{x\to a} f(x) = f(a)$.

If we use the limit law concerning quotients, we can beef up our Fact 1 to another exciting result.

Fact 2. Suppose $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials. Suppose further that $q(a) \neq 0$. Then $\lim_{x\to a} f(x) = f(a) = \dfrac{p(a)}{q(a)}$.

In the next class period we will see similar results for more classes of functions, and we will find how we can use these seemingly innocuous results to compute apparently wicked limits such as

$$\lim_{x\to 3} \frac{\sqrt{6+x} - 3}{x - 3}.$$