COURSE NOTES - 01/19/05

1. Announcements

- Grades have been posted on coursework. To date, posted grades are Quizzes 1 and 2, and Homework 1. In the future, you can always find your grades on coursework.
- Homework 2 has been posted on the course webpage. In general, homework will be posted on Friday or Saturday, and you won't receive an email notification.
- As promised, my office hours have been updated on the course webpage.
- Some folks still haven't handed in Quiz 1. The deadline for turning this quiz in is Monday, January 24, at 5pm.

2. Recap

In the limited time we had last Friday, we talked about several important topics. Most memorable where

- 5 laws of limits (finding limits of sums, differences, multiples, products, and quotients); and
- an easy way to evaluate the limit of a polynomial or rational function.

This last item in particular was fantastic. Even though $\lim_{x \to a} f(x)$ is not related to $f(a)$ in general, this little gem tells us that when $f(x)$ is a polynomial or rational function, and when a is in the domain of f, then $\lim_{x \to a} f(x) = f(a)$. In today's class we'll see that there are zillions of functions besides polynomials and rational functions which share this very nice property, and we'll use this to our advantage to evaluate certain scary looking limits.

3. Continuity, Part I

Definition. We say that a function $f(x)$ is continuous at a value a provided $\lim_{x \to a} f(x) = f(a)$.

There are a few things to notice about the definition of continuity.

(1) Continuity is a property of a function at a particular input.
(2) A function can only be continuous on its domain, since the definition of continuity at a involves $f(a)$.
(3) If f is continuous at a, then $\lim_{x \to a} f(x)$ exists.

Continuous functions are incredibly nice, and we'll see examples very soon. First, though, a few definitions related to continuous functions.

Definition. A function $f(x)$ is said to be continuous from the left at a value a provided $\lim_{x\ toa^-} f(x) = f(a)$. Similarly, $f(x)$ is continuous from the right at a if $\lim_{x\to a^+} f(x) = f(a)$. We say a function is continuous if it is continuous at all points in its domain, and we say a function is continuous on an interval $[a, b]$ if it is continuous at all values in the interval (in particular, it must be continuous form the right at a and continuous from the left at b).

Happily, continuous functions are all around us! In fact, most functions which we write down algebraically (as opposed to those we draw) are continuous on their domain. These include

Type of Function	Examples
Polynomials	$f_1(x) = x^2$ $f_2(x) = x^{17} - x^{14} + x - 2$
Rational functions	$f(x) = \dfrac{x^13 - x^5 + x}{x^3 + x^2 + x + 1}$
Exponentials	$f_1(x) = 2^x$ $f_2(x) = e^x$
Logarithms	$f(x) = \log_3(x)$
Trigonometric	$f_1(x) = \sin(x)$ $f_2(x) = \tan(x)$
Algebraic	$f(x) = \sqrt[6]{x^7 + x - 1}$
Absolute Value	$f(x) = \|x\|$

We also have ways of putting together two continuous functions to make new continuous functions:

(1) A sum of continuous functions is continuous.
(2) A difference of continuous functions is continuous.
(3) A scalar multiple of continuous functions is continuous.
(4) A product of continuous functions is continuous.
(5) A quotient of continuous functions is continuous.
(6) A composition of continuous functions is continuous.

Just to emphasize, all of these statements are statements on the domain of the function in question.

Example. The function

$$f(x) = e^{\sqrt{\sin x}}$$

is continuous on its domain, since it is the composition of continuous functions $e^x, \sqrt{x}$, and $\sin x$. In particular, we have

$$\lim_{x\to\pi/2} f(x) = f(\pi/2) = e^{\sqrt{\sin(\pi/2)}} = e^{\sqrt{1}} = e^1 = e.$$

$\square$

Example. Consider

$$g(x) = \frac{-x^7 + x^4 + x^3 + 11}{x^4 + x^3 + x^2 + x + 1}.$$

What is the value of $\lim_{x\to 1} g(x)$? Since the 1 is in the domain of $g(x)$, which is continuous, we have

$$\lim_{x\to 1} g(x) = g(1) = \frac{-1+1+1+11}{1+1+1+1+1} = \frac{12}{5}.$$

$\square$

Non-example. What is the limit

$$\lim_{x\to 1} \frac{x^2-1}{x-1}?$$

We will see how to evaluate limits like this by the end of class, but notice that 1 is not in the domain of the function, and so we can't evalute it easily by just plugging in 1 to the function.

$\square$

Another example of a problem like this can be found on the handout: Problem 3.

4. Evaluating limits of the form $\lim_{x\to a} f(x)/g(x)$

We will almost always be interested in evaluating limits of functions of the form $\dfrac{f(x)}{g(x)}$. In this section, we'll figure out how to attack problems like $\lim_{x\to a} \dfrac{f(x)}{g(x)}$. *We will be assuming throughout that* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exist.* There are 3 big cases we'll run in to:

A. $\lim_{x\to a} g(x) \neq 0$.

In this case, we can use our limit laws to evaluate:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$

If we're in the happy case where f and g are continuous at a, then this simplifies to $\dfrac{f(a)}{g(a)}$. Since this type is pretty easy, it is our favorite.

B. $\lim_{x\to a} g(x) = 0$, and $\lim_{x\to a} f(x) \neq 0$.

What happens in this case? As $x \to a$, values in the denominator are becoming tinier and tinier, while values in the numerator are approaching some fixed, nonzero number. What happens when we take some nonzero number and divide by itty bitty things? Dividing by something very small is the same as multiplying by something that's gigantic, so we're getting the product of some nonzero number with something whoppingly huge. This produces a whoppingly huge number, so the values of $f(x)/g(x)$ approach infinity as $x \to a$. We write

$$\lim_{x\to a} \frac{f(x)}{g(x)} \text{ diverges.}$$

If you want to be more specific, you might say if the function is approaching $+\infty$ (if values are becoming wildly huge and positive) or $-\infty$ (if values are becoming wildly huge and negative). If you want to be less specific, you could just say that $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ does not exist.

Example. The function $f(x) = (x-1)^{-1}$ has

$$\lim_{x\to 1^+} f(x) \text{ diverges to } +\infty$$

and

$$\lim_{x\to 1^-} f(x) \text{ diverges to } -\infty.$$

If one tries to evaluate $\lim_{x\to 1} f(x)$, one could say this limit does not exist, or that it diverges (notice that we could not describe whether it's diverging to $+\infty$ or $-\infty$ since it approaches different values from the left and right).

$\square$

C. $\lim_{x\to a} g(x) = \lim_{x\to a} f(x) = 0$.

Of all things horrible in the world (and there are a lot), few are worse than $\dfrac{0}{0}$. This symbol makes no sense at all, so in this case we have to play around with f and g in hopes of simplifying the expression so we can evaluate the limits more easily. Let's work a few examples to get a feel for this.

Example. Consider $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$. The numerator and denominator both approach 0 as $x \to 1$, so we're in this unfornuate case C. Notice, however, that

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1}(x+1) \lim_{x\to 1} \frac{x-1}{x-1} = \lim_{x\to 1}(x+1).$$

The second to last equality follows from our limits laws, which say the limit of a product of functions (in this case, the functions $f(x) = x+1$ and $g(x) = (x-1)/(x-1)$) is the product of the limit of each function. In particular, our function $g(x) = (x-1)/(x-1)$ is 1 everywhere except at the value $x = 1$, where it is undefined. Despite this, $\lim_{x\to 1} \dfrac{x-1}{x-1} = 1$, and so we get the last equality. Now to evaluate $\lim_{x\to 1}(x+1)$, we notice that $x+1$ is a polynomial, and hence continuous everywhere. So, $\lim_{x\to 1}(x+1) = 1+1 = 2$. In the end, we've shown

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} x+1 = 1+1 = 2.$$

$\square$

Example. Consider

$$\lim_{t\to 2} \frac{\sqrt{121-(t-2)}-11}{t-2}.$$

Notice that as $t \to 2$, the numerator and the denominator both approach 0. So, we have to somehow simplify the expression of our function in order to evaluate. We'll multiply by 1 in a clever way, then simplify

$$\begin{aligned}
\lim_{t\to 2}\frac{\sqrt{121-(t-2)}-11}{t-2} &= \lim_{t\to 2}\frac{\sqrt{121-(t-2)}-11}{t-2}\cdot\frac{\sqrt{121-(t-2)}+11}{\sqrt{121-(t-2)}+11}\\
&= \lim_{t\to 2}\frac{\left(\sqrt{121-(t-2)}-11\right)\left(\sqrt{121-(t-2)}+11\right)}{(t-2)\left(\sqrt{121-(t-2)}+11\right)}\\
&= \lim_{t\to 2}\frac{121-(t-2)-11^2}{(t-2)\left(\sqrt{121-(t-2)}+11\right)}\\
&= \lim_{t\to 2}\frac{-(t-2)}{(t-2)\left(\sqrt{121-(t-2)}+11\right)}\\
&= \lim_{t\to 2}\frac{-1}{\left(\sqrt{121-(t-2)}+11\right)}
\end{aligned}$$

To evaluate this last limit, notice it is a rational function which has 2 in its domain! This means we can just plug in the value 2 to evaluate. When the dust settles, we've shown

$$\lim_{t\to 2}\frac{\sqrt{121-(t-2)}-11}{t-2} = \lim_{t\to 2}\frac{-1}{\left(\sqrt{121-(t-2)}+11\right)} = -\frac{1}{22}.$$

□

More examples of limits like this can be found on the handout: Problems 1,2,4, and 5.

5. Intermediate Value Theorem

Continuous functions are wonderful for lots of reasons, not just because evaluating limits can be made simpler with them. Another nice property of continuous functions is the following.

Intermediate Value Theorem. Suppose $f(x)$ is continuous on the closed interval $[a, b]$, and suppose further that one of the following holds

- $f(a) < 0$ and $f(b) > 0$, or
- $f(a) > 0$ and $f(b) < 0$.

Then there exists some $c \in (a, b)$ with $f(c) = 0$.

Problem 1. Let $f(x) = x^3 - x^2 + x$. Show there exists a number with $f(x) = 10$.

Solution. Let $g(x) = x^3 - x^2 + x - 10$. Then $f(x) = 10$ if and only if $g(x) = 0$, so we will find a solution to $g(x) = 0$ and declare ourselves done. Now $g(x)$ is a polynomial, and hence continuous everywhere. Further, computations show that

$$g(0) = -10 < 0 \qquad \text{and} \qquad g(3) = 11 > 0.$$

Hence, since $g(x)$ is continuous on $[0, 3]$, $g(0) < 0$, and $g(3) > 0$, the intermediate value thoerem provides some $c \in (0, 3)$ with $g(c) = 0$. □

Another examplex of this kind of problem can be found on the handout: Problem 7.

6. Quiz 3 on Friday

Topics you need to know for quiz 3:

- the limit laws we discussed last Friday
- how to evaluate limits of functions, especially in the form $\lim_{x\to a} \dfrac{f(x)}{g(x)}$
- the intermediate value theorem.