COURSE NOTES - 01/24/05

1. Announcements

- We have a test coming up on Monday, January 31. The scheduled exam time is 7-9pm, and the test will be given in room 380-X. If you have a conflict with the scheduled test time, send me an email. We'll set up an alternate time.
- There will be a practice exam posted (hopefully by the afternoon of Monday, Jan 24). The practice test will be similar in format to the actual exam, and should be roughly comparable in difficulty and time. My suggestion is that you treat the practice exam as a regular exam: give yourself 2 consecutive hours to work through as much of the exam as possible. Afterwards, check your answers against the key and see which concepts or types of problems you might want to continue working on.
- There will be a review session Sunday, January 30, starting at about 6:30. Pizza will be provided for those who hunger for more than just knowledge. The location will be announced later.
- The CA for this class, Ken Chan, has asked to move his Friday office hours to Tuesday, 3:30-5:30. This change will take place this week.
- If for some reason you don't pick up a quiz or homework in class, old quizzes and homeworks will be kept in a box outside my office door. You can pick these up at your leisure.

2. Recap

Last class period we discussed a selection of homework problems. The previous class period we

- discussed the concept of continuity,
- gained a real proficiency in evaluating limits of the form $\lim_{x\to a}\left[\dfrac{f(x)}{g(x)}\right]$, and
- met the intermediate value theorem.

These topics finish off our discussion of limits, and allow us to jump back in to investigating the tangent problem we talked about a week or so ago.

3. The tangent problem

Recall that for a function $f(x)$ and a point $(a, f(a))$ on the graph of f, the tangent problem asks for the equation of the line tangent to $f(x)$ at the point $(a, f(a))$. Since we want to find the equation of a line, we must find

(1) its slope and
(2) a point on the line.

We are already provided with a point on the line (namely, $(a, f(a))$), so we only have to find the slope of this magical tangent line.

What was our idea for finding this slope? We slap another point Q on the graph of $f(x)$, say $Q = (b, f(b))$. The slope of the secant line passing through P and Q is then given by

$$\frac{f(b) - f(a)}{b - a}.$$

To find the slope of the tangent line, we allow the point Q to approach the point P we're interested in. The secant line should approach the tangent line, so that the slope of the tangent line is given by the limit of the slope of the secant line passing through P and Q as $Q \to P$. In other works

$$\boxed{\text{the slope of the line tangent to } f \text{ at } (a, f(a)) \text{ is } \lim_{b\to a} \frac{f(b) - f(a)}{b - a}.}$$

If you can believe it, it's this single fact that we're going to be investigating for almost the rest of the term.

4. Instantaneous slope

The slope of the tangent line to a function $f(x)$ at a point $(a, f(a))$ is often called the instantaneous slope of f at a. When we're evaluating this slope, we will often rewrite the limit slightly to simplify computations. In particular, we shall adopt the following convention:

$$\boxed{\text{the slope of the line tangent to } f \text{ at } (a, f(a)) \text{ is } \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.}$$

How is this expression different from the previously discussed limit? In a very real sense, it is not. All we have done is rename the point we called b as $a+h$. After this change of variables, you can see our two expressions for the slope of the line tangent to f at $(a, f(a))$ coincide. If you find the first definition much simpler to work with, you are welcome to use it to find the slope of a tangent line; I suggest the latter definition because limits of this second form are seemingly easier to evaluate.

5. A few problems

Handout, #1 Find the slope of the tangent line to $f(x) = x^2$ at the point $(3, 9)$.

Solution. We have seen above that the slope of the tangent line to $f(x)$ at a point $(a, f(a))$ is given by

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

In our case, $f(x) = x^2$ and $a = 3$. Hence the slope of the tangent line to $f(x)$ at $(3, 9)$ is

$$\begin{aligned}\lim_{h\to 0} \frac{f(3+h) - f(3)}{h} &= \lim_{h\to 0} \frac{(3+h)^2 - (3)^2}{h} = \lim_{h\to 0} \frac{9 + 6h + h^2 - 9}{h} \\ &= \lim_{h\to 0} \frac{6h + h^2}{h} = \lim_{h\to 0} \frac{h(6+h)}{h} = \lim_{h\to 0} [6 + h] = 6.\end{aligned}$$

$\square$

Handout, #2 What is the equation of the tangent line passing through $(1,5)$ on the graph $f(x) = x^3+x+3$?

Solution. To find the equation of the tangent line we must know the slope of the line and a point on the line. Since we are given a point on the line already, we only need to hunt for the slope of the tangent line. As before, we recall that the slope of the tangent line to $f(x)$ at a point $(a, f(a))$ is given by

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

In this problem, the role of $f(x)$ is played by x^3+x+3, and a is 1. Hence the slope of the tangent line to $f(x)$ at $(1,5)$ is

$$\begin{aligned}
\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0} \frac{[(1+h)^3+(1+h)+3]-[1+1+3]}{h} \\
&= \lim_{h\to 0} \frac{1+3h+3h^2+h^3+1+h+3-5}{h} \\
&= \lim_{h\to 0} \frac{4h+3h^2+h^3}{h} = \lim_{h\to 0} \frac{h(4+3h+h^2)}{h} \\
&= \lim_{h\to 0} \left[4+3h+h^2\right] = 4.
\end{aligned}$$

Now that we have the slope of the line (i.e., 4) and a point on the line (viz., $(1,5)$), we can give the equation for the tangent line using the point/slope formula: if a line has slope m and point (x_0, y_0) lies is on the line, then the equation for the line is given by $y - y_0 = m(x - x_0)$. In our case, we have

$$y - 5 = 4(x-1).$$

□

Handout, #3 What does the limit $\lim_{h\to 0} \dfrac{\frac{1}{x+h} - \frac{1}{x}}{h}$ represent? Evaluate.

Solution. Notice that if we define $f(x) = \dfrac{1}{x}$, then the limit above is

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

This limit is already very familiar to us, and we recognize that this limit gives the instantaneous slope of (i.e., the slope of the line tangent to) the function $f(x)$ at a point $(x, f(x))$ on the graph of f.

We evaluate the limit as we normally would:

$$\begin{aligned}
\lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} &= \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{-h}{hx(x+h)} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}.
\end{aligned}$$

□

The nice thing about this problem is that we are now prepared to compute the instantaneous slope of $f(x)$ at any point $(x, f(x))$ we like without having to evaluate more limits. For instance, the slope at $(1, 1)$ is -1 and the slope at $(100, f(100))$ is $-\dfrac{1}{10000}$.

Handout, #4 The instantaneous slope measures something for a given function f, as one can tell by inspecting the units of measurement for the inputs and outputs of f. In this problem, for instance, input is measured in seconds and output are measured in feet. Since the instantaneous slope measures the change in outputs of f divided by the change in inputs of f, for this problem we see that the instantaneous slope is measured in feet per second. So, the instantaneous slope measures speed!

Solution. So, to answer this problem, we need to find the instantaneous slope of $f(s)$ at the point $(60, f(60))$. The definition of instantaneous slope gives this value as

$$\lim_{h\to 0} \frac{f(60+h) - f(60)}{h} = \lim_{h\to 0} \frac{\frac{1}{4}(60+h)^2 + 20(60+h) - \left[\frac{1}{4}(60)^2 - 20(60)\right]}{h}.$$

If we wanted, we could evaluate this limit. We chose not to do it in class. □

Handout, #6 Find the slope of the tangent line to $f(x) = e^x$ at the point $(1, e)$.

Solution. We just plug our new function f and point a into the definition:

$$\begin{aligned}\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h\to 0} \frac{e^{1+h} - e^1}{h} = \lim_{h\to 0} \frac{e^1 e^h - e^1}{h} \\ &= \lim_{h\to 0} \frac{e^1(e^h - 1)}{h} = \lim_{h\to 0} e^1 \cdot \lim_{h\to 0} \frac{e^h - 1}{h} \\ &= \lim_{h\to 0} e^1 = e^1 = e.\end{aligned}$$

The fourth-to-last equality follows since we are given a product of two functions, where we know the limit of each function as $h \to 0$ exists. □