# COURSE NOTES - 01/26/05

## 1. Announcements

- As a reminder, there will be a review session Sunday, January 30, starting at about 6:30. There will also be free pizza.

## 2. Recap

We did so much great stuff last class period! Using all the machinery we've been building up for a couple weeks, we were finally able to solve the tangent problem. Highlights included:

- discovering the slope of the line tangent to $f$ at $a$ as a limit
$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h},$$
- finding the equation for the line tangent to $f$ at $a$
- realizing that when $f$ is a position function, the slope of the tangent line can be interpreted as the speed of the object being measured
- practicing problems involving tangent lines

Solving the tangent problem puts us above many, many pre-Newtonian mathematicians who weren't able to solve the tangent problem! This means we're wickedly smart.

## 3. You say poh-tay-toh...

**Definition.** The slope of the tangent line to $f$ at $a$ is called *the derivative of $f$ at $a$*, and is usually written $f'(a)$. This means
$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

With this new notation we can easily given the solution to the tangent problem. Remember last time we said if we could compute the slope of the tangent line, we would have all the information we needed to compute the equation of the line tangent to $f$ at $a$. In particular, if this slope is $m$, then the equation was given by
$$y-f(a)=m(x-a).$$
With out new notation, this means the solution to the tangent problem is
$$\boxed{y-f(a)=f'(a)(x-a).}$$

Handout, #1 Find the equation of the tangent line to $y = \sqrt{x}$ at $x = 4$.

*Solution.* Using the notation above, we have $a = 4$ and $f(a) = \sqrt{4} = 2$. To get the equation of the tangent line, we only have to computer $f'(4)$. We do this by the definition

$$\begin{aligned} f'(4) &= \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h} \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} = \lim_{h\to 0} \frac{4+h-4}{h(\sqrt{4+h}+2)} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{4+h}+2)} = \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} = \frac{1}{4}. \end{aligned}$$

Hence, the equation of the line tangent to $f$ at 4 is

$$y - 2 = \frac{1}{4}(x - 4).$$

□

Handout, #2 Find the equation of the line tangent to the graph of $y = x^{\frac{1}{2}}$ at $x = 1$.

*Solution.* Note that $f(x) = \frac{1}{\sqrt{x}}$. Also we have $a = 1$ and $f(a) = 1/\sqrt{1} = 1$. So, we only have to compute $f'(1)$, which we do using the definition.

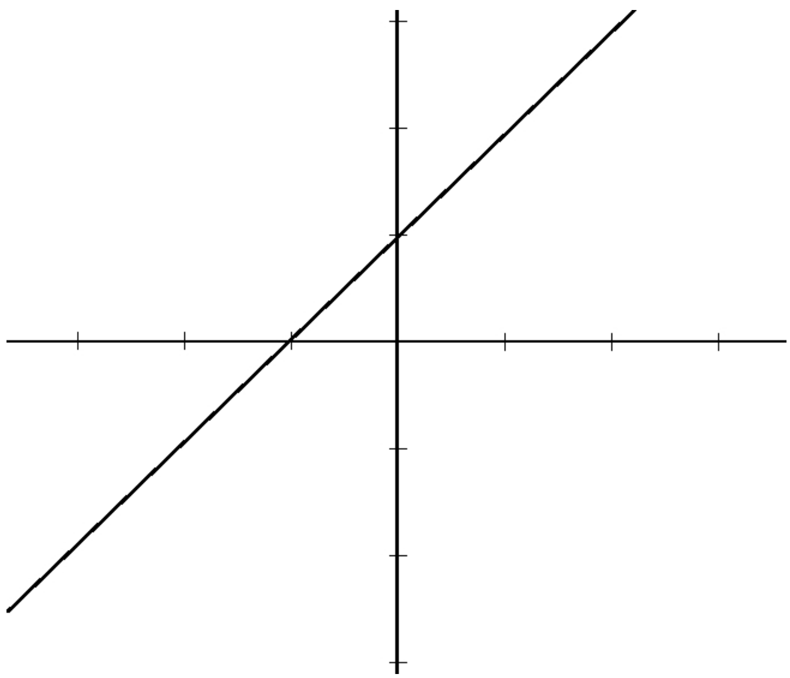
$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{1+h}} - 1}{h} \\ &= \lim_{h\to 0} \frac{\frac{1-\sqrt{1+h}}{\sqrt{1+h}}}{h} = \lim_{h\to 0} \frac{1 - \sqrt{1+h}}{h\sqrt{1+h}} \frac{1+\sqrt{1+h}}{1+\sqrt{1+h}} \\ &= \lim_{h\to 0} \frac{1 - (1+h)}{h\sqrt{1+h}(1+\sqrt{1+h})} \\ &= \lim_{h\to 0} \frac{-h}{h\sqrt{1+h}(1+\sqrt{1+h})} = \lim_{h\to 0} \frac{-1}{\sqrt{1+h}(1+\sqrt{1+h})} = -\frac{1}{2}. \end{aligned}$$

□

## 4. The geometry of the derivative

Since the derivative of a function at a point $a$ gives the slope of the line tangent to $f$ at $a$, we can get a lot of information about the derivative $f'(a)$ at various values of $a$ by sketching the graph of $f(x)$.

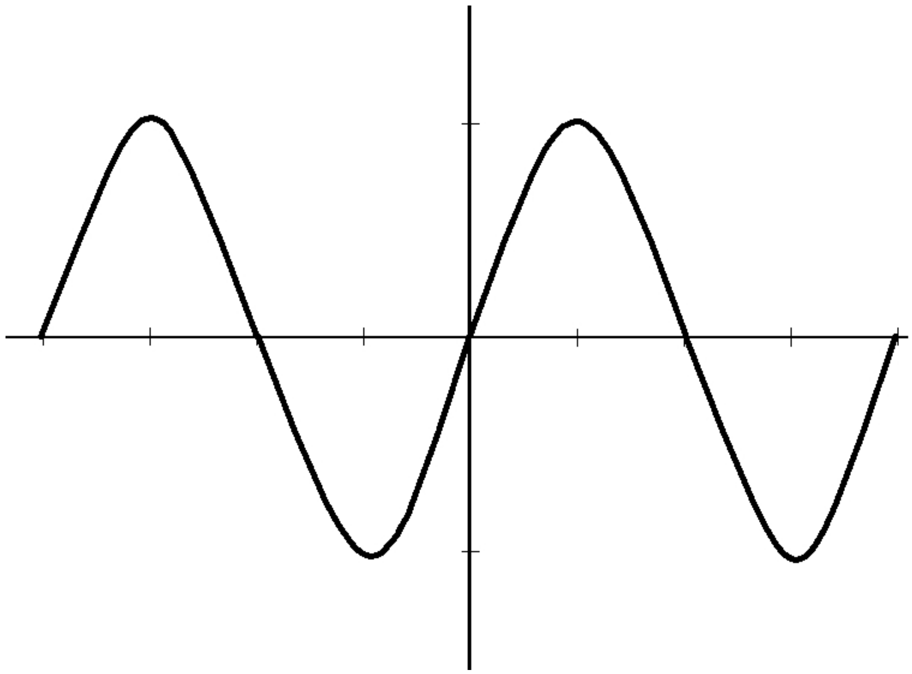
**Example.** Suppose $f(x)$ is a line with slope 1 as shown below. What is $f'(a)$ for any value $a$?



*Solution.* If we sketch the line tangent to $f$ at any value of $a$, we see that this tangent line is just the line we started off with! Since $f'(a)$ is the slope of the line tangent to $f$ at $a$, and since the tangent line is the same as the original line, this means that $f'(a) = 1$ for all values of $a$.

If our original line has slope 3 instead of 1, then $f'(a) = 3$ for all values of $a$. $\square$

Handout, #3 The graph of a function $f(x)$ is given below.



(3a) Find all values of $a$ such that $f'(a) = 0$.

*Solution.* Since $f'(a)$ is the slope of the line tangent to $f$ at $a$, points where $f'(a) = 0$ are those points where the tangent line to $f$ at $a$ has slope 0. Lines with slope 0 are horizontal, so to answer this problem, we need to find values of $a$ such that the tangent line to $f$ at $a$ is horizontal. From inspection, we see these values are $-3, -1, 1, 3$. $\square$

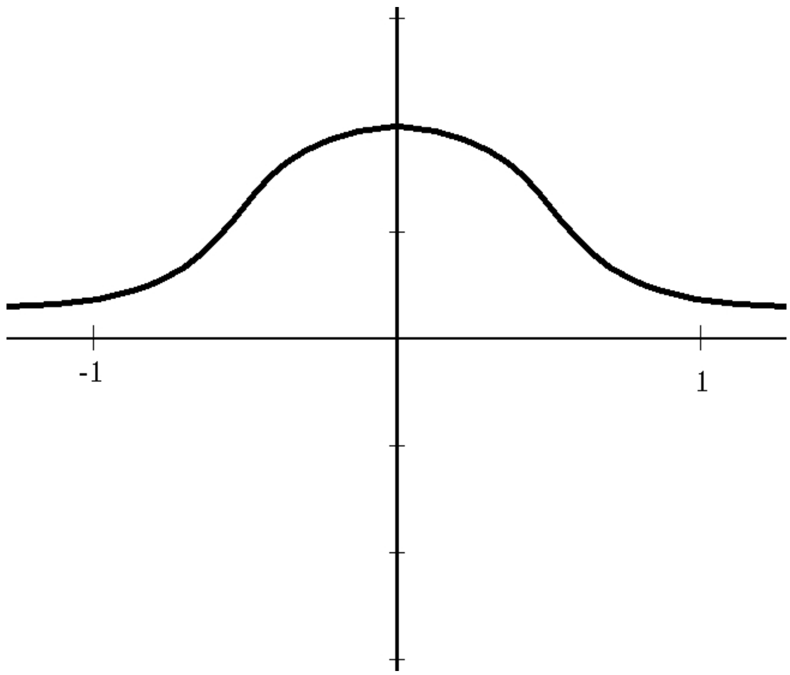
(3b) Find intervals over which $f'(a) > 0$.

*Solution.* Again, since $f'(a)$ is the slope of the line tangent to $f$ at $a$, we are looking for values of $a$ for which the tangent line to $f$ at $a$ is positive. By sketching the tangent line at various points, we can see this happens on the intervals $[-4, -3), (-1, 1)$, and $(3, 4]$. $\square$

(3c) Find intervals over which $f'(a) < 0$.

*Solution.* This time we are looking for values of $a$ for which the tangent line to $f$ at $a$ is negative. By sketching the tangent line at various points, we can see this happens on the intervals $(-3, -1), (1, 3)$. $\square$

Handout, #4 The graph of a function $g(x)$ is given below.




(4d) Order the following set of numbers in increasing order: $g'(-1), g'(1), g'(-1)$.

*Solution.* If we sketch the tangent lines at $-1, 0$, and $1$, we can see that

- $g'(-1) > 0$
- $g'(0) = 0$
- $g'(1) < 0$

Hence we have

$$g'(1) < g'(0) < g'(-1).$$

□

(4b) For what value of $a$ is $g'(a)$ largest?

*Solution.* From above, we see there are point $a$ with $g'(a) > 0$ and points of $a$ with $g'(a) < 0$. Hence $g'(a)$ is largest when the slope of the tangent line to $g$ at $a$ is positive and as steep as possible. From the graph, it seems the tangent line is steepest (with positive slope) when $a$ is about $-\frac{1}{2}$. □

(4c) For what value of $a$ is $g'(a)$ smallest?

*Solution.* Again, we see there are point $a$ with $g'(a) > 0$ and points of $a$ with $g'(a) < 0$. Hence $g'(a)$ is smallest when the slope of the tangent line to $g$ at $a$ is negative and as steep as possible. From the graph, it seems the tangent line is steepest (with negative slope) when $a$ is about $\frac{1}{2}$. $\square$

## 5. The derivative as a rate of change

We'll talk about this more in the future, but it's important to notice that the derivative gives us information about how the outputs of a function change with respect to a change in the inputs of a function. In particular, since

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

we see that the derivative in fact gives us how outputs of a function change with respect to very small changes in the input of the function.

The example we saw in class last time was the case when $f$ is a position function. We saw in this case $f'(a)$ measures the instantaneous speed of the object at time $a$. Question 5 on your handout is about a function $C(x)$ whose output measures total profit (in dollars) and whose input is measured in units of books produced (in thousands). Hence $C'(a)$ gives the information of how profit per book changes when production is slightly different from $a$ (one might say how profit per book changes with the production of 1 more book).