COURSE NOTES - 01/30/05

1. Announcements

- Another reminder: this Sunday, Jan. 30, we'll be having a review session. It will start at 6:30 and will be held in room 383-N.

2. Recap

Last class period we started to use some terminology that's going to carry through the rest of the quarter, though many of the ideas are ones we've been using for weeks now. Important topics were:

- the definition of the derivative of a function f at a point a, written $f'(a)$
- estimating the derivative using the graph of a function
- interpreting the meaning of the derivative for 'special' functions, such as when $f(x)$ is a position function.

3. Linearization

Soon we're going to dive deeply into the world of derivatives, but today we're going to spend the short class time we have together to discuss an application of tangent lines.

There are plenty of functions that we can evaluate easily. For instance, evaluating polynomials is not very difficult, since they only involve operations like addition, subtraction, and multiplication. For the same reason, computing the value of rational functions is also relatively easy. But is it easy to compute the value of functions like $\sqrt{x}$ or $\log_2(x)$?

Functions like $\sqrt{x}$, $\log_2(x)$, and $\sin(x)$ are functions that we see all the time in calculus. Fortunately, there are some values of these functions we can compute easily. For instance, we all know that $\sqrt{4} = 2$ or that $\sin\left(\frac{\pi}{2}\right) = 1$. But could you easily compute $\sqrt{4.01}$? Without a calculator, this would be quite difficult to do. However, we can use the fact that the tangent line to f at a point a is a pretty good approximation of the function f 'near' a. Hence, if we can compute the equation of the tangent to f at a, we can use this to approximate the value of $f(a + \varepsilon)$ for small values of ε.

Handout, #1 Let $f(x) = \sqrt{x}$.

(a) Find the equation of the line tangent to f at $x = 4$.

Solution. We know that the universal solution to the tangent problem for a function f at a point a is

$$y - f(a) = f'(a)(x - a),$$

so we need to compute $f(4)$ and $f'(4)$. It's easy to see that $f(4) = \sqrt{4} = 2$, and we also have

$$f'(4) = \lim_{h\to 0}\frac{f(4+h)-f(4)}{h} = \lim_{h\to 0}\frac{\sqrt{4+h}-\sqrt{4}}{h}$$
$$= \lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} = \lim_{h\to 0}\frac{4+h-4}{h(\sqrt{4+h}+2)}$$
$$= \lim_{h\to 0}\frac{h}{h(\sqrt{4+h}+2)} = \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2} = \frac{1}{4}.$$

Hence, the equation of the line tangent to f at 4 is

$$y-2 = \frac{1}{4}(x-4),$$

or if you prefer

$$y = \frac{1}{4}(x-4)+2.$$

$\square$

(b) Use this line to approximate the values of $\sqrt{4.01}$ and $\sqrt{4.1}$. Which is a better approximation?

Solution. To approximate $f(4.01) = \sqrt{4.01}$, I will plug the value $x = 4.01$ into the equation of the tangent line we found above. Hence, I have

$$\sqrt{4.01} \approx \frac{1}{4}(4.01-4)+2 = 2\frac{1}{400}.$$

Similarly,

$$\sqrt{4.1} \approx \frac{1}{4}(4.1-4)+2 = 2\frac{1}{40}.$$

The approximation using this method is—generally speaking—better for values closer to the point $(a, f(a))$ we started off with. In this case, we should therefore expect that the approximate value for $\sqrt{4.01}$ is better than that for $\sqrt{4.1}$. $\square$

Handout, #2 Let $f(x) = \sin(x)$.

(a) Use the graph of f to determine $f'(\frac{\pi}{2})$, then give the equation of the line tangent to f at $(\frac{\pi}{2}, 1)$.

Solution. Since we know what the graph of $sin(x)$ looks like, we can see that we expect $f'(\frac{pi}{2}) = 0$. Using our universal solution to the tangent problem, we see that the tangent line has the equation

$$y-1 = 0(x-\frac{\pi}{2}),$$

or in other words

$$y = 1.$$

$\square$

(b) Use the previous result to approximate $\sin(\frac{\pi}{2}+\varepsilon)$, where ε is an arbitrary real number. Is the approximation better for large or small values of ε.

Solution. Without doing anything else, we can already answer that we expect the approximation to be better for values close to $\frac{\pi}{2}$. This means the approximation is generally better for small values of ε.
Now for the appximation. Again, instead of evaluating the function at $\frac{\pi}{2} + \varepsilon$, we will plug this number instead into the equation for the tangent line. This gives

$$\sin(\frac{\pi}{2} + \varepsilon) \approx 1,$$

since the tangent line is the constant function $y = 1$. □