COURSE NOTES - 02/02/05

1. Recap

In our last 'real' class period, we talked about linearization. Linearization is an application of tangent lines which allow us to approximate the value of a function $f(x)$ near a point a when we can find the tangent line to f at a. We usually use this technique for function $f(x)$ which are difficult to evaluate (like $f(x) = \sin(x)$ or $g(x) = \sqrt{x}$).

2. The derivative as a function

So far we have considered the derivative of a function *at a fixed point* a. Today, we change our perspectice slightly and instead speak of the derivative as a function itself. For a given input x, this derivative function will return for us the derivative of f at the point x. Explicitly

Definition. For a function $f(x)$, the derivative of f, written either $f'(x)$ or $\frac{d}{dx}[f(x)]$, is the function defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Notice that in particular this derivative function is very much like the limits we have been considering in the past few weeks. The only difference is that instead of having an explicit value for x in the limit above, we leave this quantity as a variable.

Handout, #1 Using only that that derivative $f'(a)$ is the slope of the tangent line to f at a,

(a) find the derivative of $f(x) = c$, where $c \in \mathbb{R}$.

Solution. The graph of the function $f(x) = c$ is a horizontal line which passes through the point $(0, c)$. What is the tangent to this graph at a point on the graph? One can see this tangent line is another horizontal line, which of course has slope 0. This means that for any point x we want, the slope of the tangent line through f at x is 0. In other words, for all x we have $f'(x) = 0$.
We could also compute this derivative using the definition. Let's try it out.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} \\ &= \lim_{h \to 0} \frac{0}{h} = 0. \end{aligned}$$

□

(b) find the derivative of $f(x) = mx + b$.

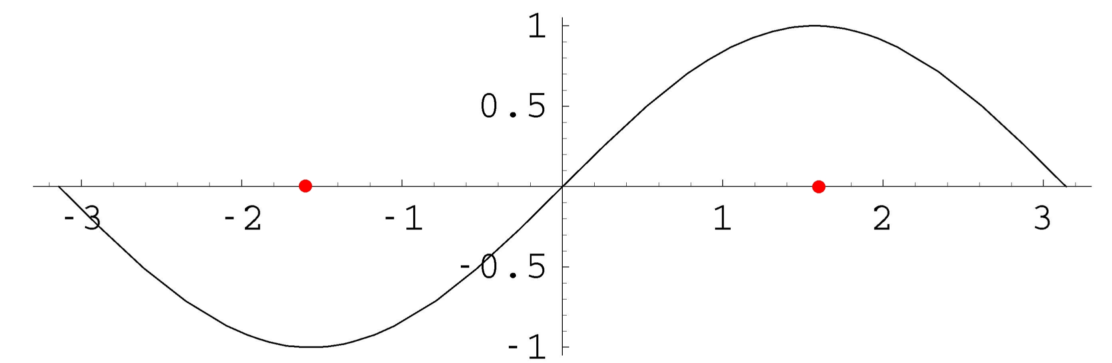
Solution. If we sketch the graph of the line $f(x) = mx + b$ and consider the slope of the tangent line to f at a point x, we see that the tangent line is the line $f(x)$ itself! This means that the slope of the tangent line at a point x (which we know is $f'(x)$) is the slope of the original line (which we know is m). This gives $f'(x) = m$.
Again, if we wanted to we could compute this limit using the definition. Let's do it:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{m(x+h) + b - (mx+b)}{h} \\ &= \lim_{h\to 0} \frac{mx + mh + b - mx - b}{h} = \lim_{h\to 0} \frac{mh}{h} = \lim_{h\to 0} m = m. \end{aligned}$$

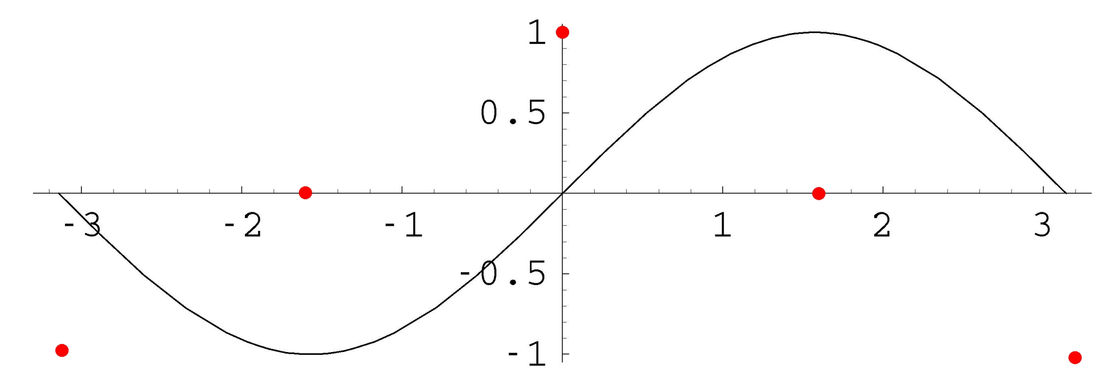
□

(c) use the graph of $f(x) = \sin(x)$ to sketch $f'(x)$; does this function look familiar?

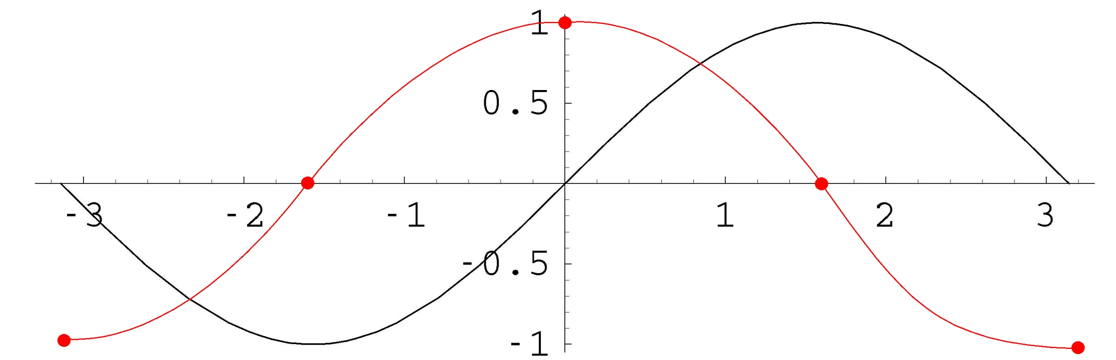
Solution. Using the graph of $\sin(x)$, we see first that $f'(x)$ is 0 for $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$. Hence, we can plot these values on the graph of $f'(x)$ straightaway.



What might we try next? If we look at $x = 0$, it appears that the slope of the tangent line is 1. (To see this, try sketching the line $y = x$ onto this graph. It should appear tangent to the graph at 0.) Hence we can believe $f'(0) = 1$. Similarly, we can 'eyeball' that the slope of the tangent line through $x = -\pi$ is -1 and that the slope of the tangent line through $x = \pi$ is also -1. So, we can plot $f'(0) = 1$ and $f'(-\pi) = f'(\pi) = -1$.



Using these 5 points, and noticing that the slope of the tangent stays fairly close to 1 near 0 and fairly close to -1 near $-\pi$ and π, it is not unthinkable that $f'(x)$ look something like this:



$\square$

Handout, #2 Use the definition of the derivatice to compute $f'(x)$ where

(a) $f(x) = x^2$.

Solution. Using the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0} 2x+h = 2x. \end{aligned}$$

Note: Computing the derivative in this way is great, because it allows us to quickly compute $f'(a)$ for any value of a a person might give us. So, for instance, if you were asked to find the slope of the tangent line to f at the points $-1, 0, 1, 2, 3,$ and 4, you could do so without evaluating 6 separate limits. Simply compute $f'(x)$ first, then plug $-1, 0, 1, 2, 3,$ and 4 into $f'(x)$ (in this case, we would get $f'(-1) = -2$, $f'(0) = 0$, $f'(1) = 2$, $f'(2) = 4$, $f'(3) = 6$, and $f'(4) = 8$). $\square$

(b) $f(x) = \sqrt{x}$.

Solution. Using the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

Notice that in the last step we're assuming $x \neq 0$. $\square$

(c) $f(x) = \sqrt[3]{x}$.

Solution. So far we've only done a handful of different types of limits. This one is a bit different, and requires a new 'trick.' You won't be expected to improvise this kind of trick, so don't be too scared by this example. I just want to show you we can do different kinds of derivatives "with our bare hands." Ok, let's do it:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}\\
&= \lim_{h\to 0}\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}\,\frac{\sqrt[3]{(x+h)^2}+\sqrt[3]{x(x+h)}+\sqrt[3]{x^2}}{\sqrt[3]{(x+h)^2}+\sqrt[3]{x(x+h)}+\sqrt[3]{x^2}}\\
&= \lim_{h\to 0}\frac{\sqrt[3]{(x+h)^3}+\sqrt[3]{x(x+h)^2}+\sqrt[3]{x^2(x+h)}-\sqrt[3]{x(x+h)^2}-\sqrt[3]{x^2(x+h)}-\sqrt[3]{x^3}}{h(\sqrt[3]{(x+h)^2}+\sqrt[3]{x(x+h)}+\sqrt[3]{x^2})}\\
&= \lim_{h\to 0}\frac{x+h-x}{h(\sqrt[3]{(x+h)^2}+\sqrt[3]{x(x+h)}+\sqrt[3]{x^2})} = \lim_{h\to 0}\frac{h}{h(\sqrt[3]{(x+h)^2}+\sqrt[3]{x(x+h)}+\sqrt[3]{x^2})}\\
&= \lim h \to 0\frac{1}{\sqrt[3]{(x+h)^2}+\sqrt[3]{x(x+h)}+\sqrt[3]{x^2}} = \frac{1}{\sqrt[3]{x^2}+\sqrt[3]{x^2}+\sqrt[3]{x^2}} = \frac{1}{3\sqrt[3]{x^2}}.
\end{aligned}$$

□

Handout, #3 Use

- $\sin(x+h) = \sin(x)\cos(h)+\cos(x)\sin(h)$
- $\lim_{h\to 0}\dfrac{\sin(h)}{h} = 1$
- $\lim_{h\to 0}\dfrac{\cos(h)-1}{h} = 0$

to compute $\dfrac{d}{dx}[\sin(x)]$.

Solution. Again, this example won't use any spectacular ideas except for the facts above. It is our job to massage to put our limit in a form where we can use these facts. Anyway, let's proceed

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} = \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h}\\
&= \lim_{h\to 0}\left(\frac{\sin(x)(\cos(h)-1)}{h}+\frac{\cos(x)\sin(h)}{h}\right)\\
&= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)}{h}+\lim_{h\to 0}\frac{\cos(x)\sin(h)}{h}\\
&= \left[\lim_{h\to 0}\sin(x)\right]\left[\lim_{h\to 0}\frac{\cos(h)-1}{h}\right]+\left[\lim_{h\to 0}\cos(x)\right]\left[\lim_{h\to 0}\frac{\sin(h)}{h}\right]\\
&= \sin(x)\cdot 0+\cos(x)\cdot 1 = \cos(x)
\end{aligned}$$

□