COURSE NOTES - 02/02/05

1. Recap

In the last class period, we defined the derivative of a function. We saw that the derivative is defined by the limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h},$$

and at a point a the value of the derivative gives the slope of the tangent line to f at a. We also saw how one might graph the derivative of a function when given the graph of the function itself.

2. Non-differentiability

We say that a function f is differentiable at a if the derivative of f is defined at a. That is to say, f is differentiable at a if the following limit exists:

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Although we will be dealing mostly with differentiable functions in this class, there are a handful of times we will encounter functions which have points which are not differentiable. Although there are many ways a function could fail to be continuous at a point a, there are three typical types of non-differentiability.

(1) **Kinks.** A function is not differentiable at a point where the graph of f has a kink or corner (one might also call such a point on the graph a cusp). Essentially, these places fail to be differentiable because the left and right hand limits

$$\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$$

do not match up. For instance, the absolute value function $f(x) = |x|$ fails to be differentiable at 0 because

$$\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} = -1$$

and

$$\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} = 1.$$

(2) **Discontinuities.** A function is not differentiable at a point where the graph of f is not continuous. In this case, the limit

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

does not exist, because the denominator approaches 0 as $h \to 0$, while the numerator approaches some finite, nonzero number (remember that since f is not differentiable at a, we have

$$\lim_{h\to 0} f(x) \neq f(a),$$

and hence $\lim_{h\to 0} f(a+h) - f(a) \neq 0$).

(3) **Vertical Tangents.** Finally, a function is not differentiable at a point on the graph where the tangent line to f is a vertical line. This is because the slope of the tangent to the graph at this point is infinite, which in our class corresponds to 'does not exist.'

3. Second derivative

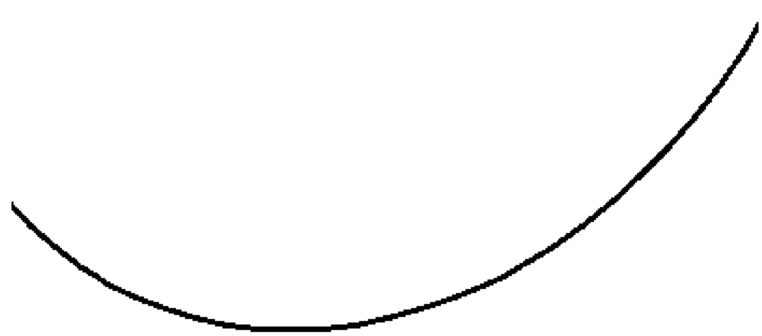
Last class we introduced the derivative of a function f, which is itself a function. Why not take the derivative of this function? Why not indeed!

Definition. The second derivative of a function, written $f''(x)$, is $\dfrac{d}{dx}[f'(x)]$.

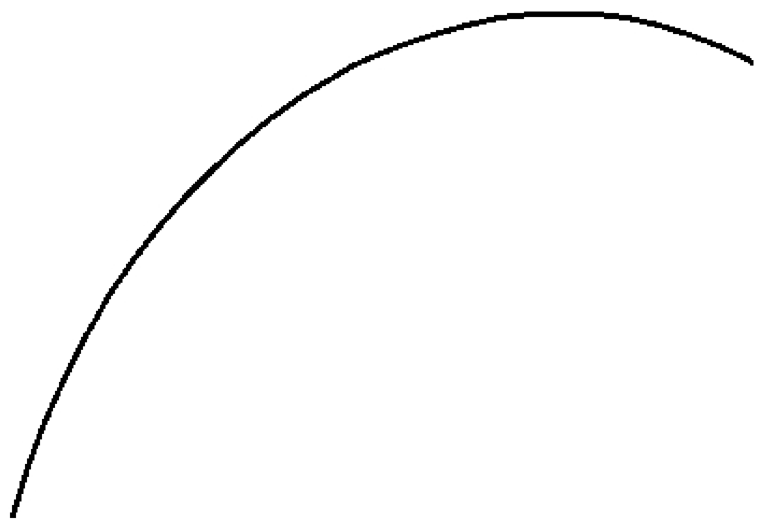
You will not be surprised to hear that one can also speak of the third derivative of a function, or even the fourth or fifth. The pariticularly enthusiastic might wonder if a function can have infinitely many derivatives. However, these more exotic creatures aren't going to be so useful to us in this class. But what about the second derivative? The second derivative of a function has a few nice applications.

(1) If f is a position function, then we have already seen that $f'(x)$ is the function which gives the instantaneous speed of the object being measured. And just as the first derivative measures the change in the function values given a small change in inputs, the second derivative measures the change in the derivative values (i.e., slopes of tangent lines) given a small change in inputs. Since a $f'(x)$ is speed for a position function f, this means that $f''(x)$ measures how speed is changing with respect to time. In other words, $f''(x)$ measures the acceleration of the object being measured by f.

(2) For an arbitrary function f, a similar analysis holds. $f''(x)$ measures how the slopes of tangents change given a small changes in inputs. Hence, if $f''(x) > 0$, this means that the slopes of tangent lines are *increasing* as we move from left to right. Such places on the graph are said to be concave up, and generally they look like the following picture.



On the other hand, if $f''(x) < 0$, this means that the slopes of tangent lines are *decreasing* as we move from left to right. Such places on the graph are said to be concave down, and they look like this.



Places where $f''(x) = 0$ are places where the slopes of tangent lines are constant. These places are called inflexion points of the graph f. The point $(0, 0)$ on the curve $y = x^3$ is an inflexion point. In fact, one can see that the graph of x^3 is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, \infty)$ (sketch the graph of x^3 and see for yourself!).

(3) Concavity is also useful for our new skills in linearization. In particular, we can use concavity to detect if our linear approximations are over or underestimates. Notice for the picture of concave down above, tangent lines lie above the graph of the function. This means linear approximations taken from these tangent lines will be overestimates. Similarly, tangent lines lie below the graph of a function where the graph is concave up. This means that linear approximations taken from these tangent lines will be underestimates. In other words,
- if $f''(a) > 0$, then linear approximations near a will be underestimates, and
- if $f''(a) < 0$, then linear approximations near a will be overestimates.