COURSE NOTES - 02/07/05

1. Recap

Last class period we talked about two main topics

- Places on a function $f(x)$ where f is not differentiable. The three bigies were: kinks (or corners) of the graph of f, discontinuities of f, and points on the graph with a vertical tangent line.
- The second derivative of $f(x)$. In particular, we
  - gave a definition for the second derivative;
  - saw that for position functions it measures acceleration;
  - saw that in general it measures the concavity of $f(x)$;
  - discussed inflexion points; and
  - used it to determine when linear approximations are over- or underestimates.

2. Properties of the derivative

There are a few properties of the derivative which we will use over and over and over again. These rules are so natural that you might find yourself using them without even thinking.

Theorem. If f and g are differentiable functions and $c \in \mathbb{R}$, then

- $\dfrac{d}{dx}\left[cf(x)\right] = c\dfrac{d}{dx}\left[f(x)\right]$
- $\dfrac{d}{dx}\left[f(x)+g(x)\right] = \dfrac{d}{dx}\left[f(x)\right] + \dfrac{d}{dx}\left[g(x)\right]$
- $\dfrac{d}{dx}\left[f(x)-g(x)\right] = \dfrac{d}{dx}\left[f(x)\right] - \dfrac{d}{dx}\left[g(x)\right]$.

These facts can be really useful in evaluating derivatives. For instance, one of your homework problems asked you to compute

$$\frac{d}{dx}\left[x+\sqrt{x}\right].$$

You saw in your homework that evaluating this limit required that you be fairly careful in your computations. On the other hand, evaluating $\frac{d}{dx}\left[x\right]$ and $\frac{d}{dx}\left[\sqrt{x}\right]$ are each fairly standard, so we can find $\frac{d}{dx}\left[x+\sqrt{x}\right]$ by computing the sum of these two easier limits.

3. Power Rule

Until now, to compute derivatives we have had to rely on the definition of the derivative. As you have learned, this is not a painless process, since you are required to evaluate certain tricky limits. Happily, there are rules that make computing certian derivatives quite easily. The best example of such a rule is the Power rule.

Let's review some results we have already seen:

$$\begin{array}{ll} \frac{d}{dx}[c] = 0 & \frac{d}{dx}[mx] = m \\ \frac{d}{dx}[x^2] = 2x & \frac{d}{dx}[x^3] = 3x^2 \\ \frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}} & \frac{d}{dx}[\sqrt[3]{x}] = \frac{1}{3\sqrt[3]{x^2}} \end{array}$$

Each of these derivatives fits in to a pattern known as the Power Rule.

Power Rule. $\frac{d}{dx}[x^n] = nx^{n-1}$

Obviously, this rule is a dramatic simplification of the definition of the derivative we have been using. For instance, we can compute $\frac{d}{dx}[x + \sqrt{x}]$ quickly as

$$\frac{d}{dx}\left[x + \sqrt{x}\right] = \frac{d}{dx}\left[x^1 + x^{\frac{1}{2}}\right] = 1 \cdot x^{1-1} + \frac{1}{2}x^{\frac{1}{2}-1} = 1 + \frac{1}{2}x^{-\frac{1}{2}}.$$

Similarly, we can easily compute otherwise scary derivatives, like

$$\frac{d}{dx}\left[x^{10} + \sqrt[3]{x} + x^{1000} + \frac{1}{x^2} + \sqrt{x}\right] = 10 \cdot x^9 + \frac{1}{3}x^{\frac{1}{3}-1} + 1000 \cdot x^{999} - 2x^{-3} + \frac{1}{2}x^{\frac{1}{2}-1}.$$

Clearly, the power rule is a powerful tool. You'll be using it a lot in the future.

4. Exponentials

With the addition of the power rule to our toolbox, we are now able to compute the derivative of any polynomial we like, along with more than a handful of algebraic functions (like $\sqrt[5]{x}$). We also have all the equipment we need to compute the derivative of many exponential functions. Let's try it out:

$$\begin{aligned} \frac{d}{dx}a^x &= \lim_{h\to 0}\frac{a^{x+h} - a^x}{h} = \lim_{h\to 0}\frac{a^x a^h - a^x}{h} \\ &= \lim_{h\to 0} a^x \frac{a^h - 1}{h} = a^x \lim_{h\to 0}\frac{a^h - 1}{h}. \end{aligned}$$

As it happens, this limit is a number we have already discussed: $\lim_{h\to 0}\frac{a^h - 1}{h} = \log a$ (here, as always in this class, log is the natural logarithm—i.e., with base e). Hence, we have

$$\frac{d}{dx}[a^x] = (\log a)\, a^x.$$

In particular, when $a = e$ we see that
$$\frac{d}{dx}\left[e^x\right] = (\log e)\, e^x = e^x$$
since $\log e = 1$.

Putting this together with the power rule, we can compute derivatives like
$$\frac{d}{dx}\left[e^x + a^x + \sqrt{x} + \frac{1}{x}\right]$$
with surprisingly little difficulty. In fact,
$$\frac{d}{dx}\left[e^x + a^x + \sqrt{x} + \frac{1}{x}\right] = e^x + (\log a)\, a^x + \frac{1}{2}x^{-\frac{1}{2}} - x^{-2}.$$