# COURSE NOTES - 02/09/05

## 1. Announcements

- On Friday we have our 6th quiz of the term, which marks our entry into the second half of the class. If you feel that your grade isn't where you want it to be, it's time to step it up!
- For Friday's homework, you may omit 2.10.21(b)

## 2. Recap

Last class period we began to gain some real proficiency in computing derivatives. The tools we picked up were

- derivative rules for sums and differences of products, along with scaled functions;
- the power rule, which gives $\frac{d}{dx}[x^n] = n \cdot x^{n-1}$; and
- rules governing the derivative of simple exponentials (for instance, $\frac{d}{dx}[a^x] = (\log a)a^x$).

## 3. Product Rule

Last class period we saw that the derivative of a sum of functions is the sum of the derivative of each function. Is the same true for products? Let's investigate $\frac{d}{dx}[f \cdot g]$ with an example. Let's take $f(x) = g(x) = x$. Then we have

$$\frac{d}{dx}[f \cdot g] = \frac{d}{dx}\left[x^2\right] = 2x^{2-1} = 2x$$

and

$$\frac{d}{dx}[f]\frac{d}{dx}[g] = \frac{d}{dx}[x]\frac{d}{dx}[x] = 1 \cdot 1 = 1.$$

In this case, then, we have $\frac{d}{dx}[f \cdot g] \neq \frac{d}{dx}[f]\frac{d}{dx}[g]$.

So what is $\frac{d}{dx}[f \cdot g]$? We'll work from the definition:

$$\begin{aligned}
\frac{d}{dx}[f \cdot g] &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \frac{\left[f(x) + \overbrace{f(x+h) - f(x)}^{F}\right]\left[g(x) + \overbrace{g(x+h) - g(x)}^{G}\right] - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x)g(x) + f(x)G + g(x)F + FG - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x)G}{h} + \lim_{h \to 0} \frac{g(x)G}{h} + \lim_{h \to 0} \frac{FG}{h} \\
&= f(x) \underbrace{\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}}_{g'(x)} + g(x) \underbrace{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}_{f'(x)} \\
&\quad + \underbrace{\lim_{h \to 0} f(x+h) - f(x)}_{0} \underbrace{\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}}_{g'(x)} \\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

In summary, if $f$ and $g$ are differentiable functions, then

$$\boxed{\frac{d}{dx}[f \cdot g] = f(x)g'(x) + g(x)f'(x).}$$

Handout, #1 Evaluate $\frac{d}{dx}[xe^x]$.

*Solution.* With $f(x) = x$ and $g(x) = e^x$, we can evaluate this derivative using the product rule:

$$\frac{d}{dx}[xe^x] = x\frac{d}{dx}[e^x] + e^x\frac{d}{dx}[x] = xe^x + e^x = e^x(x+1).$$

□

Handout, #2 Evaluate $\frac{d}{dx}[x + xe^x]$.

*Solution.*

$$\frac{d}{dx}[x + xe^x] = \frac{d}{dx}[x] + \frac{d}{dx}[xe^x] = 1 + e^x(x+1).$$

□

Handout, #3 Evaluate $\frac{d}{dx}\left[\frac{e^x}{x}\right]$.

*Solution.*

$$\frac{d}{dx}\left[\frac{e^x}{x}\right] = \frac{d}{dx}\left[e^x x^{-1}\right] = e^x\frac{d}{dx}\left[x^{-1}\right] + x^{-1}\frac{d}{dx}[e^x] = e^x \cdot (-1)x^{-2} + x^{-1} \cdot e^x.$$

If we had a good reason to, we could simplify this answer more...but in this case, we'll just leave it as is. □

In this last problem, we evaluated the derivative of a quotient by rewriting it as a product of functions. We can use this trick to find derivatives of all sorts of quotients. So suppose we're interested in finding $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$. With $F = \frac{f}{g}$, we have $Fg = f$. Taking the derivative of each side and setting them equal to each other, we find $f'(x) = F(x)g'(x) + F'(x)g(x)$, and solving for $F'(x)$ we find

$$\begin{aligned}\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = F'(x) = \frac{f'(x) - F(x)g'(x)}{g(x)} &= \frac{\frac{g(x)}{g(x)}f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)} \\ &= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}\end{aligned}$$

This is often called the quotient rule. Notice that in the formula above, it is VERY important that the function in the numerator is called $f(x)$ and the function in the denominator is called $g(x)$; the minus sign in our expression will give us problems if we switch these names around! So be careful!

Let's get a little exercise using the quotient rule:

Handout, #4 Evaluate $\frac{d}{dx}\left[\frac{x+e^x}{x^3+x}\right]$.

*Solution.* Using the quotient rule, we have

$$\begin{aligned}\frac{d}{dx}\left[\frac{x+e^x}{x^3+x}\right] &= \frac{(x^3+x)\frac{d}{dx}\left[x+e^x\right] - (x+e^x)\frac{d}{dx}\left[x^3+x\right]}{(x^3+x)^2} \\ &= \frac{(x^3+x)\left[1+e^x\right] - (x+e^x)\left[3x^2+1\right]}{(x^3+x)^2}.\end{aligned}$$

As usual, we could simplify this answer if we wished, but there's no real need to now. □

Handout, #5 Evaluate $\frac{d}{dx}\left[\frac{\frac{2}{x}-a^x}{x^2-1}\right]$.

*Solution.* Using the quotient rule, we have

$$\begin{aligned}\frac{d}{dx}\left[\frac{\frac{2}{x}+a^x}{x^2-1}\right] &= \frac{d}{dx}\left[\frac{2x^{-1}+a^x}{x^2-1}\right] \\ &= \frac{(x^2-1)\frac{d}{dx}\left[2x^{-1}+a^x\right] - (2x^{-1}+a^x)\frac{d}{dx}\left[x^2-1\right]}{(x^2-1)^2} \\ &= \frac{(x^2-1)\left[-2x^{-2}+(\log a)a^x\right] - (2x^{-1}+a^x)\left[2x\right]}{(x^2-1)^2}.\end{aligned}$$

Again, we could simplify this answer if we wanted to, but we won't. □