COURSE NOTES - 02/09/05

1. Comment on notation

I should have made a comment concerning a standard notation for higher derivatives when we first introduced second derivatives, but I forgot. As I did mention, though, the second derivative of a function $f(x)$ is usually written $f''(x)$. You won't be surprised to see that the third derivative is often writen $f'''(x)$, though you should also become familiar with another notation for the third derivative: $f^{(3)}(x)$. The fourth derivative of a function is almost always written $f^{(4)}(x)$, and all higher derivatives use this same parenthetical notation.

2. Recap

In the last few class periods we've gained some tools to helps us compute derivatives of functions quickly. In Wednesday's class, we saw how to take the derivative of a product or quotient of functions f and g in terms of the functions and their derivatives. In particular, we saw the product rule

$$\frac{d}{dx}[f \cdot g] = f\frac{d}{dx}[g] + g\frac{d}{dx}[f]$$

and the quotient rule

$$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{g\frac{d}{dx}[f] - f\frac{d}{dx}[g]}{g^2}.$$

Today we're going to switch our focus from computing derivatives to exploring some reasons why a person might be interested in derivatives in the first place. The applications are the real reason people continue to be interested in calculus today, and in particular they are the reason that so many students are required to take a calculus class.

3. Geometry of a function and its derivatives: an overview

One of the first connections we made between the geometry of the graph of a function f and information concerning derivatives of f is that places on the graph of f which are 'flat' (i.e., where the tangent line to f at a is horizontal) correspond to solutions to $f'(x) = 0$. However, we can gain a lot of information about the geometry of the graph of $f(x)$ by using information concerning its derivatives. The following list is by no means exhaustive, but covers most of the things we've seen in this class so far.

$f(x)$ is flat at a	$\leftrightarrow$	$f'(a) = 0$
$f(x)$ is increasing on (a, b)	$\leftrightarrow$	$f'(x) > 0$ on (a, b)
$f(x)$ is decreasing on (a, b)	$\leftrightarrow$	$f'(x) < 0$ on (a, b)
$f(x)$ is concave up on (a, b)	$\leftrightarrow$	$f''(x) > 0$ on (a, b)
$f(x)$ is concave down on (a, b)	$\leftrightarrow$	$f''(x) < 0$ on (a, b)
$f'(x)$ is increasing on (a, b)	$\leftrightarrow$	$f''(x) > 0$ on (a, b)
$f'(x)$ is decreasing on (a, b)	$\leftrightarrow$	$f''(x) < 0$ on (a, b)
Slopes of tangent lines of f are increasing on (a, b)	$\leftrightarrow$	$f''(x) > 0$ on (a, b)
Slopes of tangent lines of f are decreasing on (a, b)	$\leftrightarrow$	$f''(x) < 0$ on (a, b)

4. Local Maxs and Mins

Perhaps the most widespread application of derivatives is in computing local maxima and minima of a function $f(x)$. A local maximum of a function $f(x)$ is a point a for which $f(b) < f(a)$ for all points b 'close to' a; similarly, a local minimum of a function $f(x)$ is a point a for which $f(b) > f(a)$ for all points b 'close to' a. For instance, the absolute value function as a local minimum at $x = 0$, since all points near 0 have absolute value which is positive. Also, the function $sin(x)$ has local maxima at points $\cdots, -\frac{3\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \cdots$, and local minima at points $\cdots, -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \cdots$. For a given function $f(x)$ is is quite natural to find local maxima and minima. But how will derivatives help us do that?

Try to graph a function with a local max or min (your choice) at a point $x = a$. If you study your drawing, you will see that the derivative of your function at a will be either 0 or undefined. Hence, to find places where a function *might* have a local max or min, we need to find places where $f'(x) = 0$ or $f'(x)$ is undefined.

Example. Find possible local maxs and mins of the function $f(x) = xe^x$.

Solution. We have just seen above that local max and mins can only occur if $f'(x)$ is zero or undefined. Hence to find possible local maxs and mins of $f(x)$, we need to compute its derivative first. We have seen the derivative of this function before (using the product rule), and so we recall that $f'(x) = (x + 1)e^x$.

Now possible local maxs and mins occur when $f'(x)$ is zero or undefined. Since $f'(x)$ is defined everywhere, we need only find solutions to $f'(x) = 0$, or in other words $(x + 1)e^x = 0$. Now $e^x > 0$ for any number x we plug in, so we have $f'(x) = 0$ if and only if $x + 1 = 0$. Of course, this happens if and only if $x = -1$. Hence, the possible local max/min of $f(x) = xe^x$ is the point $x = -1$. $\square$

Example. Find possible local maxs and mins of the function $f(x) = x^3$.

Solution. Again, local maxs and mins can only occur if $f'(x)$ is zero or undefined. Hence to find possible local maxs and mins of $f(x)$, we need to compute $f'(x)$. Using the power rule, we have $f'(x) = 3x^2$.

Now possible local maxs and mins occur when $f'(x)$ is zero or undefined. Since $f'(x)$ is defined everywhere, we need only find solutions to $f'(x) = 0$, or in other words $3x^2 = 0$. Since this only happens if $x = 0$, the possible local max/min of $f(x) = x^3$ is the point $x = 0$. $\square$

Notice that in each of these examples we have not said whether the possible local max/min is *actually* a local max/min. Indeed, in the second example the possible max/min is not a max/min at all. So how can we determine whether a function has, say, a local maximum at a point *without looking at its graph*? There are actually a few answers. In all cases, we are assuming that it has already been shown that $f'(a) = 0$ or is undefined.

(1) (Bare Hands) The definition of a local max says that a is a local max of f if for any point b near a, $f(a) > f(b)$. Hence, we could find a point very close to a and to its left, say b_l, and determine if $f(a) > f(b_l)$. If we also found a point b_r very close to a and to its right, we could then test if $f(a) > f(b_r)$. If both of these inequalities hold, we're golden.

The downside to this procedure is twofold. First, it isn't clear how close the points b_l and b_r need to be to a to count at 'near a.' Second, we would have to compute the actual value of $f(b_l)$ and $f(b_r)$, which can be pretty difficult without a calculator.

(2) (First derivative test) We might also examine the slopes of tangent lines near a to get a feeling for what the graph of f looks like near a. In particular, if we find a point very close to a and to its left, call it b_l, with $f'(b_l) > 0$ and another point very close to a and to its right, call it b_r, with $f'(b_r) < 0$, then we can be assured that the function has a maximum at a. Why? Sketch a line segment with positive slope whose right endpoint is a, and a line segment with negative slope whose left endpoint is a. What does it look like? Sort of like a $\bigwedge$, right? And the tip of this $\bigwedge$ is the point $(a, f(a))$. So you can see that a is a local max.

What are good parts to this approach? First, when we have to find points b_l and b_r, we can determine what 'close by' actually means using the IVT. In particular, provided our derivative is a continuous function (which it almost always is), all we need to do to choose b_l is to make sure $f'(x)$ has no zeros on the interval (b_l, a), and similarly to choose b_r we only have to make sure $f'(x)$ has no zeros on the interval (a, b_r). The second reason it is a good approach is that we don't actually have to compute $f'(b_l)$ or $f'(b_r)$. Instead, we only have to compute whether these numbers are positive or negative. In general, this is far easier to do than computing the actual derivatives at these points.

(3) (Second derivative test) We might also notice that at a local max, the function in question is often concave down. Hence, if we can show that $f''(a) < 0$, then we can immediately conclude that f has a maximum at a. The upside to this technique is that it is very quick (provided you have the second derivative). The downside is that it might happen that $f''(a) = 0$. In this situation, one cannot conclude from the second derivative alone whether f has a max or a min at a. Instead, you would likely have to use the second technique above.