COURSE NOTES - 02/14/05

1. Announcements

Your second midterm is Monday, February 28th. Again, it will be given from 7-9pm. We will have a review session Sunday, February 27th, as we did for the last midterm.

2. Recap

Last class period we took a break from computing derivatives to focus on applications of derivatives. We saw

- the connection between the geometry of the graph of $f(x)$ and properities of its derivatives;
- how to use derivatives to find possible maxima/minima of a function $f(x)$; and
- how we might use the first or second derivative to determine whether a possible maximum/minimum is actually a maximum or a minimum (or neither!).

3. More on Maxima/Minima

So how can we determine whether a function has, say, a local maximum at a point *without looking at its graph*? There are actually a few answers. **In all cases, we are assuming that it has already been shown that $f'(a) = 0$ or is undefined.**

(1) Bare Hands The definition of a local max says that a is a local max of f if for any point b near a, $f(a) > f(b)$. Hence, we could find a point very close to a and to its left, say b_l, and determine if $f(a) > f(b_l)$. If we also found a point b_r very close to a and to its right, we could then test if $f(a) > f(b_r)$. If both of these inequalities hold, we're golden.

The downside to this procedure is twofold. First, it isn't clear how close the points b_l and b_r need to be to a to count at 'near a.' Second, we would have to compute the actual value of $f(b_l)$ and $f(b_r)$, which can be pretty difficult without a calculator.

(2) First derivative test We might also examine the slopes of tangent lines near a to get a feeling for what the graph of f looks like near a. In particular, if we find a point very close to a and to its left, call it b_l, with $f'(b_l) > 0$ and another point very close to a and to its right, call it b_r, with $f'(b_r) < 0$, then we can be assured that the function has a maximum at a. Why? Sketch a line segment with positive slope whose right endpoint is a, and a line segment with negative slope whose left endpoint is a. What does it look like? Sort of like a $\bigwedge$, right? And the tip of this $\bigwedge$ is the point $(a, f(a))$. So you can see that a is a local max.

In a similar way, you can verify that

- if $f'(b_l) < 0$ and $f'(b_r) > 0$, then f has a minimum at a;

- if $f'(b_l) > 0$ and $f'(b_r) > 0$, then f has neither a max nor a min at a; and
- if $f'(b_l) < 0$ and $f'(b_l) < 0$, then f has neither a max nor a min at a.

What are good parts to this approach? First, when we have to find points b_l and b_r, we can determine what 'close by' actually means using the IVT. In particular, provided our derivative is a continuous function (which it almost always is), all we need to do to choose b_l is to make sure $f'(x)$ has no zeros on the interval (b_l, a), and similarly to choose b_r we only have to make sure $f'(x)$ has no zeros on the interval (a, b_r). The second reason it is a good approach is that we don't actually have to compute $f'(b_l)$ or $f'(b_r)$. Instead, we only have to compute whether these numbers are positive or negative. In general, this is far easier to do than computing the actual derivatives at these points. Finally, this technique is great because it works every time.

(3) <u>Second derivative test</u> We might also notice that at a local max, the function in question is often concave down. Hence, if we can show that $f''(a) < 0$, then we can immediately conclude that f has a maximum at a. The upside to this technique is that it is very quick (provided you have the second derivative). The downside is that it might happen that $f''(a) = 0$. In this situation, one cannot conclude from the second derivative alone whether f has a max or a min at a. Instead, you would use the second technique above.

Example. Find all maxima and minima of the function $f(x) = x^3$.

Solution. For convenience later, we note that $f'(x) = 3x^2$ and $f''(x) = 6x$. Now potential maxima and minima occur where $f'(x) = 0$ or is undefined. Since $3x^2$ is defined everywhere, we only need to determine when $3x^2 = 0$. Of course, this happens only when $x = 0$, so our list of potential max/mins for f is only $x = 0$. Is it a maximum or a minimum? If we try the second derivative test, we see that $f''(0) = 0$, and we cannot draw a conclusion. Hence, we'll try the first derivative test. To choose the points b_l and b_r, we only have to be careful that there aren't zeroes of $f'(x)$ in the intervals $(b_l, 0)$ and $(0, b_r)$. Since $f'(x) = 0$ has only one solution (namely, $x = 0$), we choose $b_l = -1$ and $b_r = 1$. Now $f'(-1) = 3(-1)^2 > 0$, and also $f'(1) = 3(1)^2 > 0$. By the first derivative test, we see that f has neither a max nor a min at $x = 0$.

In conclusion, $f(x)$ has no relative maxima or minima. □

Example. Find all maxima and minima of the function $f(x) = x^4$.

Solution. For convenience later, we note that $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Since potential maxima and minima occur where $f'(x) = 0$ or is undefined, we see that our list of potential max/mins for f is $x = 0$. Is it a maximum or a minimum? If we try the second derivative test, we see that $f''(0) = 0$, and we cannot draw a conclusion. For the first derivative test, in choosing the points b_l and b_r we only have to be careful that there aren't zeroes of $f'(x)$ in the intervals $(b_l, 0)$ and $(0, b_r)$. Since $f'(x) = 0$ has only one solution (namely, $x = 0$), we choose $b_l = -1$ and $b_r = 1$. Now $f'(-1) = 4(-1)^3 < 0$, and $f'(1) = 4(1)^3 > 0$. By the first derivative test, we see that f has a min at $x = 0$.

In conclusion, $f(x)$ has exactly one local minimum (occuring at $x = 0$) and no local maxima. □

Example. Find all maxima and minima of the function $f(x) = xe^x$.

Solution. We have seen before that $f'(x) = (x+1)e^x$. Now

$$f''(x) = \frac{d}{dx}\left[(x+1)e^x\right] = (x+1)\frac{d}{dx}\left[e^x\right] + e^x\frac{d}{dx}\left[x+1\right] = (x+2)e^x.$$

Now $f'(x)$ is defined everywhere, so that potential maxima and minima of the function occur where $f'(x) = 0$. Since $e^x > 0$ for any x, we see that $f'(x) = 0$ only when $x = -1$. Is it a maximum or a minimum? Let's try the second derivative test, we see that $f''(-1) = e^{-1}(-1+2)$. Again, $e^{-1} > 0$ and also $-1+2 > 0$, so that $f''(-1) > 0$. We conclude that f has a minimum at $x = -1$.

In conclusion, $f(x)$ has exactly one local minimum (occuring at $x = -1$) and no local maxima. □

4. Derivatives of Trigonometric functions

We're going to move back towards computing derivatives for the next week or so. Today we're going to review our results on derivatives of trigonometric functions.

Recall in particular that a few weeks ago we showed that

$$\frac{d}{dx}\left[\sin(x)\right] = \cos(x).$$

This required a trigonometric identity involving sine and leaned on some limits like $\lim_{h\to 0}\frac{\sin(h)}{h}$. If you're so inclined, you can look at a geometric justification of those limits in your book (I believe it's in Section 3.4).

One can use the same techniques to compute

$$\frac{d}{dx}\left[\cos(x)\right] = -\sin(x).$$

These two derivatives lie at the heart of all derivatives of trigonometric functions.

Example. Compute $\frac{d}{dx}\left[\tan(x)\right]$.

Solution. Since $\tan(x) = \frac{\sin(x)}{\cos(x)}$, we use the quotient rule:

$$\frac{d}{dx}\left[\tan(x)\right] = \frac{\cos(x)\frac{d}{dx}\left[\sin(x)\right] - \sin(x)\frac{d}{dx}\left[\cos(x)\right]}{\left(\cos(x)\right)^2} = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)}$$
$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x).$$

□

One could use a similar technique to show that $\frac{d}{dx}\left[\sec(x)\right] = \sec(x)\tan(x)$.

At the very least, you should remember these four derivatives (sine, cosine, tangent, and secant). Most important, you MUST know the derivatives of sine and cosine like the back of your hand.