COURSE NOTES - 02/16/05

1. Recap

Last time we finished our discussion of local maxima/minima of a function and their relationship to the derivative of the function. In particular we discussed the so-called first and second derivative tests for determining when a potential max/min of a function is either a maximum or minimum (or neither). We spent the second half of class talking about derivatives of trigonometric functions.

2. The Chain Rule

Today we talk about our last tool for computing derivatives of functions. In the past we have found quick rules for computing derivatives of polynomials, exponentials, trigonometrics, products, and quotients. Today, we will discuss how one takes the derivative of a composition of functions. This will give us the technology to compute just about any derivative we like.

The chain rule tells us that for differentiable functions $f(x)$ and $g(x)$,

$$\frac{d}{dx}[f \circ g] = f'(g(x)) \cdot g'(x).$$

In words, the derivative of a composition is the derivative of the outer function evaluated at the inner function times the derivative of the inner function. We could discuss how a person goes about proving such a claim, but instead let's work lots of examples.

Handout, #8 Compute $\frac{d}{dx}[\sin(2x)]$.

Solution. Since $\sin(2x) = \sin(x) \circ (2x)$, the chain rule gives

$$\begin{aligned} \frac{d}{dx}[\sin(2x)] &= \frac{d}{dx}[\sin(x) \circ (2x)] = \left(\frac{d}{dx}[\sin(x)] \circ (2x)\right) \cdot \frac{d}{dx}[2x] \\ &= (\cos(x) \circ (2x)) \cdot (2) = 2\cos(2x). \end{aligned}$$

$\square$

Handout, #9 Compute $\frac{d}{dx}\left[e^{x^2+1}\right]$.

Solution. Since $e^{x^2+1} = e^x \circ (x^2+1)$, the chain rule gives

$$\begin{aligned} \frac{d}{dx}\left[e^{x^2+1}\right] &= \frac{d}{dx}\left[e^x \circ (x^2+1)\right] = \left(\frac{d}{dx}[e^x] \circ (x^2+1)\right) \cdot \frac{d}{dx}\left[x^2+1\right] \\ &= \left(e^x \circ (x^2+1)\right) \cdot (2x) = 2xe^{x^2+1}. \end{aligned}$$

$\square$

Handout, #10 Compute $\frac{d}{dx}[\sin(\cos(x))]$.

Solution. Since $\sin(\cos(x)) = \sin(x) \circ \cos(x)$, the chain rule gives

$$\begin{aligned}\frac{d}{dx}[\sin(\cos(x))] &= \frac{d}{dx}[\sin(x) \circ \cos(x)] = \left(\frac{d}{dx}[\sin(x)] \circ \cos(x)\right) \cdot \frac{d}{dx}[\cos(x)] \\ &= (\cos(x) \circ \cos(x)) \cdot (-\sin(x)) = -\sin(x)\cos(\cos(x)).\end{aligned}$$

□

Handout, #11 Compute $\frac{d}{dx}[(f(x))^n]$.

Solution. Since $(f(x))^n = x^n \circ f(x)$, the chain rule gives

$$\begin{aligned}\frac{d}{dx}[(f(x))^n] &= \frac{d}{dx}[x^n \circ f(x)] = \left(\frac{d}{dx}[x^n] \circ f(x)\right) \cdot \frac{d}{dx}[f(x)] \\ &= \left(nx^{n-1} \circ f(x)\right) \cdot (f'(x)) = nf'(x)(f(x))^{n-1}.\end{aligned}$$

□

Handout, #12 Compute $\frac{d}{dx}\left[e^{\sqrt{x^2+1}}\right]$.

Solution. Since $e^{\sqrt{x^2+1}} = e^x \circ \sqrt{x^2+1}$, the chain rule gives

$$\begin{aligned}\frac{d}{dx}\left[e^{\sqrt{x^2+1}}\right] &= \frac{d}{dx}\left[e^x \circ \sqrt{x^2+1}\right] = \left(\frac{d}{dx}[e^x] \circ \sqrt{x^2+1}\right) \cdot \frac{d}{dx}\left[\sqrt{x^2+1}\right] \\ &= \left(e^x \circ \sqrt{x^2+1}\right) \cdot \left(\frac{d}{dx}\left[\sqrt{x^2+1}\right]\right).\end{aligned}$$

But what is $\frac{d}{dx}\left[\sqrt{x^2+1}\right]$? Again, we'll use the chain rule:

$$\begin{aligned}\frac{d}{dx}\left[\sqrt{x^2+1}\right] &= \frac{d}{dx}\left[\sqrt{x} \circ (x^2+1)\right] = \left(\frac{d}{dx}\left[x^{\frac{1}{2}}\right] \circ (x^2+1)\right) \cdot \frac{d}{dx}[x^2+1] \\ &= \left(\frac{1}{2}x^{-\frac{1}{2}} \circ (x^2+1)\right) \cdot (2x) = x(x^2+1)^{-\frac{1}{2}}.\end{aligned}$$

Hence, continuing our original computation:

$$\frac{d}{dx}\left[e^{\sqrt{x^2+1}}\right] = e^{\sqrt{x^2+1}} \cdot \left(x(x^2+1)^{-\frac{1}{2}}\right).$$

□

Handout, #13 Use the fact that $x = e^{\log(x)}$ to compute $\frac{d}{dx}[\log(x)]$.

Solution. Taking the derivative of the left and right hand side of the equation above, we see that

$$1 = \frac{d}{dx}[x] = \frac{d}{dx}\left[e^{\log(x)}\right].$$

Now

$$\frac{d}{dx}\left[e^{\log(x)}\right] = \left(\frac{d}{dx}[e^x] \circ \log(x)\right) \cdot \frac{d}{dx}[\log(x)] = e^{\log(x)}\frac{d}{dx}[\log(x)] = x\frac{d}{dx}[\log(x)].$$

Hence, we have

$$1 = x\frac{d}{dx}[\log(x)],$$

and so

$$\frac{d}{dx}[\log(x)] = \frac{1}{x}.$$

□

3. A comment on notation

We have already see that the derivative of a function $y = f(x)$ can be written either as $f'(x)$ or $\frac{d}{dx}[f(x)]$. People might also write this derivative as $\frac{df}{dx}$ or as $\frac{dy}{dx}$. Be thee 'ware.

4. A max/min problem

I wanted to do one extra example of a non-trivial max/min problem so you could see how a person solves such a problem.

Handout, #1 Find the local maximum and minimum values of the function $f(x) = x^3 + \frac{3}{2}x^2 - 18x + 1$.

Solution. We know that max/min values occur where $f'(x) = 0$ or where the derivative is undefined. Since $f'(x) = 3x^2 + 3x - 18$, the derivative is defined everywhere. Hence, potential max/mins occur when $f'(x) = 0$. Since $f'(x) = 3(x^2 + x - 6) = 3(x + 3)(x - 2)$, our potential max/min values are $x = -3, 2$.

To determine whether -3 is a maximum or a minimum, we can use either the first or second derivative test. Let's use the first derivative test. This means I need to find a point 'close' and to the left of -3 and a point 'close' and to the right of -3, then examine the sign of $f'(x)$ at each of these points. Recall that 'close' just means that $f'(x)$ has no zeros between the point we choose and -3. So, I choose my left point at -4 and my right point at 0. Notice that $f'(-4) = 3(-4 + 3)(-4 - 2) > 0$, and that $f'(0) = 3(3)(-2) < 0$. Using the first derivative test, this tells us f has a MAXIMUM at $x = -3$.

To determine if 2 is a maximum or a minimum, let's again use the first derivative test. Again, I need points to the left and right of 2 that are 'nearby,' and then I need to investigate the sign of $f'(x)$ at each of these points. For my left point I'll choose 0, and on the right I'll choose 3. Now we've already seen $f'(0) < 0$, and it's not hard to see that $f'(3) > 0$. Using the first derivative test, this gives that f has a MINIMUM at $x = -2$. □