COURSE NOTES - 02/18/05

1. Some examples

In the last two weeks we have developed a handful of results that have enormous power. The power rule and rules for differentiating exponentials and trigonometric functions are the foundation of this power, and we can bootstrap these results to nearly any function we like using the product, quotient, and chain rules. Today I'll work through a few examples to show you just how powerful our techniques are, and you'll spend about 10 minutes computing derivatives yourselves.

Handout, #1 Compute $\frac{d}{dx}\left[e^{e^x}\right]$.

Solution. We see that the function e^{e^x} can be written as the composition $e^x \circ e^x$. Hence we can compute this derivative using the chain rule, together with our knowledge of differentiating simple exponential functions.

$$\begin{aligned}\frac{d}{dx}\left[e^{e^x}\right] &= \frac{d}{dx}\left[e^x \circ e^x\right] = \left(\frac{d}{dx}\left[e^x\right] \circ e^x\right)\frac{d}{dx}\left[e^x\right] \\ &= \left(e^x \circ e^x\right)e^x = e^x e^{e^x}.\end{aligned}$$

□

Handout, #2 Compute $\frac{d}{dx}\left[x^x\right]$. (Note: the function x^x is only well behaved for $x > 0$, so we'll only consider x^x on this domain).

Solution. We need to rewrite x^x into some product/composition of exponentials/polynomials before we can use our rules for differentiating. Now since $e^{\log x} = x$ on the domain $x > 0$, we can write $x^x = \left(e^{\log(x)}\right)^x = e^{x\log(x)}$. Hence we can compute this derivative using the chain rule, together with our knowledge of differentiating simple exponential, logarithmic, and polynomial functions.

$$\begin{aligned}\frac{d}{dx}\left[x^x\right] &= \frac{d}{dx}\left[e^{xlog(x)}\right] = \frac{d}{dx}\left[e^x \circ (x\log(x))\right] \\ &= \left(\frac{d}{dx}\left[e^x\right] \circ (x\log(x))\right)\frac{d}{dx}\left[x\log(x)\right] = \left(e^x \circ (x\log(x))\right)\left(x\frac{d}{dx}\left[\log(x)\right] + \log(x)\frac{d}{dx}\left[x\right]\right) \\ &= \left(e^{x\log(x)}\right)\left(x\frac{1}{x} + \log(x)\right) = \left(x^x\right)\left(1+\log(x)\right).\end{aligned}$$

□

Handout, #3 Compute $\dfrac{d}{dx}[\log(\log(x))]$.

Solution. In this case, $\log(\log(x))$ is a simple composition: $\log(x) \circ \log(x)$. So we differentiate using the chain rule.

$$\begin{aligned}\frac{d}{dx}[\log(\log(x))] &= \frac{d}{dx}[\log(x) \circ \log(x)] \\ &= \left(\frac{d}{dx}[\log(x)] \circ \log(x)\right) \frac{d}{dx}[\log(x)] = \left(\frac{1}{x} \circ \log(x)\right) \frac{1}{x} \\ &= \left(\frac{1}{\log(x)}\right) \frac{1}{x} = \frac{1}{x\log(x)}.\end{aligned}$$

$\square$

Handout, #4 Compute $\dfrac{d}{dx}\left[\sqrt[3]{x^3+\sin(x)}\right]$.

Solution. Again, we have a simple composition of functions, so we use the chain rule.

$$\begin{aligned}\frac{d}{dx}\left[\sqrt[3]{x^3+\sin(x)}\right] &= \frac{d}{dx}\left[x^{\frac{1}{3}} \circ (x^3+\sin(x))\right] \\ &= \left(\frac{d}{dx}\left[x^{\frac{1}{3}}\right] \circ (x^3+\sin(x))\right) \frac{d}{dx}\left[x^3+\sin(x)\right] \\ &= \left(\frac{1}{3}x^{-\frac{2}{3}} \circ (x^3+\sin(x))\right)(3x^2+\cos(x)) \\ &= \frac{1}{3}(x^3+\sin(x))^{-\frac{2}{3}}(3x^2+\cos(x)).\end{aligned}$$

$\square$