# COURSE NOTES - 02/23/05

## 1. Announcements

This coming Monday (February 28th) we have a midterm from 7pm to 9pm. To prepare for the midterm,

- I have posted a practice exam on the course webpage. The format of the midterm will closely follow the format of this practice exam, and the level of difficulty should be comparable. My suggestion is that you carve out a 2 hour block of time to take the practice midterm without notes or your book.
- We will have a review session Sunday (February 27th) as we did before the last midterm. The review session will be held in 383-N, the same room we used for the last review session. The time is not yet finalized...we'll figure that out in class on Friday.

## 2. Recap

In the last class period we computed many derivatives using all the technology we've built up over the last 2 or 3 weeks. In this class period we'll add another feather to our cap by discussing derivatives of inverse trig functions and learning implicit differentiation.

## 3. Derivatives of inverse trig functions

We can use the technique we developed for solving $\frac{d}{dx}[\log(x)]$ to solve for the derivative of the inverse of almost any function we like. An important class of inverse functions are the inverse trig functions, which include (but aren't limited to) $\arcsin(x)$, $\arccos(x)$, and $\arctan(x)$. We're interested in the derivatives of any of these functions, but today we'll discuss

$$\frac{d}{dx}[\arcsin(x)] \qquad \text{and} \qquad \frac{d}{dx}[\arctan(x)].$$

**Example.** Compute $\frac{d}{dx}[\arcsin(x)]$.

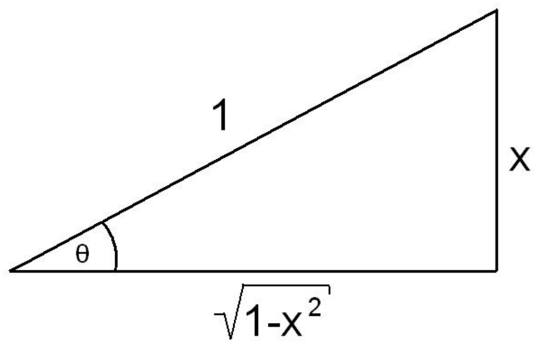
*Solution.* The defining equation for $\arcsin(x)$ is

$$\sin(\arcsin(x)) = x.$$

The derivative of the right hand side is 1, and the derivative of the left hand side is

$$\frac{d}{dx}[\sin(x) \circ \arcsin(x)] = \left(\frac{d}{dx}[\sin(x)] \circ \arcsin(x)\right) \frac{d}{dx}[\arcsin(x)] = \cos(\arcsin(x)) \frac{d}{dx}[\arcsin(x)].$$

Hence we have $\frac{d}{dx}[\arcsin(x)] = \frac{1}{\cos(\arcsin(x))}$. But what is $\cos(\arcsin(x))$? Consider the picture below.




Notice that $\theta = \arcsin(x)$, since $\sin(\theta) = x$. Also the length of the bottom edge is $\sqrt{1-x^2}$ using the Pythagorean theorem. We can see then that $\cos(\arcsin(x)) = \cos(\theta) = \sqrt{1-x^2}$. Hence, we have shown that

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}.$$

□

We could use the same type of technique to show

$$\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}.$$

## 4. Implicit differentiation

To this point, we have gained serious proficiency in evaluating the derivative of a function $y = f(x)$. However, it often happens that we are given some relationship between an independent variable $x$ and some dependent variable $y$. For instance, the equation for the unit circle

$$x^2 + y^2 = 1$$

show that $y$ is dependent on $x$, but that $y$ itself is not a function of $x$.

In these types of scenarios, we will still be interested in how $y$ changes given a small change in $x$. That is to say, we will still be interested in the derivative $\frac{dy}{dx}$. How can we go about finding this quantity?

Given an equation $f(x, y) = 0$, we might try solving $y$ in terms of $x$, and then computing derivatives. For instance, in the case of the circle we can see that $y = \pm\sqrt{1-x^2}$, and we can evaluate the derivative $\frac{dy}{dx}$ using the techniques from the last several weeks.

But what if $x$ and $y$ are related by an equation like $x^3 + y^3 = 6xy$? Solving for $y$ in terms of $x$ in this case will be very difficult. Worse still, there are equations relating $y$ and $x$ where we *won't* be able to solve for $y$ in terms of $x$, no matter how hard we try or how clever we are. So what can we do if we want to find $\frac{dy}{dx}$? The idea is to take the given equation relating $x$ and $y$ and whack it with the derivative operator $\frac{d}{dx}$ on both sides. In taking these derivatives, we will find that terms of the form $\frac{dy}{dx}$ continually pop out of the expressions, allowing us to solve for $\frac{dy}{dx}$. Let's do a few examples to get a feel for this.

**Example.** Suppose that $y^2 = 4x$. Find $\frac{dy}{dx}$.

*Solution.* In this case, we could find $\frac{dy}{dx}$ by solving for $y$ and evaluating the derivative in the traditional way. But let's try our other technique instead. So, we differentiate each side of the equation $y^2 = 4x$ and attempt to solve for $\frac{dy}{dx}$. Now $\frac{d}{dx}[4x] = 4$. What is $\frac{d}{dx}[y^2]$? Now the key point is $y^2 = x^2 \circ y$, so we have

$$\frac{d}{dx}\left[y^2\right] = \frac{d}{dx}\left[x^2 \circ y\right] = \left(\frac{d}{dx}\left[x^2\right] \circ y\right)\frac{d}{dx}[y] = 2y\frac{dy}{dx}.$$

Combining this with the derivative of the right hand side as computed before, we have

$$4 = 2y\frac{dy}{dx}.$$

Solving for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx} = \frac{2}{y}.$$

$\square$

**Example.** Suppose that $x^3 + y^3 = 6xy$. Find $\frac{dy}{dx}$.

*Solution.* Again, our technique is to apply $\frac{d}{dx}$ to both sides of the equation relating $x$ and $y$, then solve for $\frac{dy}{dx}$. Now the left hand side has derivative

$$\frac{d}{dx}\left[x^3 + y^3\right] = 3x^2 + 3y^2\frac{dy}{dx}$$

and the derivative of the right hand side is

$$\frac{d}{dx}[6xy] = 6\left(x\frac{dy}{dx} + y\right).$$

This gives the equation $3x^2 + 3y^2\frac{dy}{dx} = 6x\frac{dy}{dx} + 6y$, and we solve

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}.$$

$\square$