COURSE NOTES - 03/02/05

1. Announcements

(1) The scale for Midterm 2 is as follows
- A $\rightarrow [91, 100]$
- B $\rightarrow [83, 91)$
- C $\rightarrow [70, 83)$
- D $\rightarrow [62, 70)$
- F $\rightarrow [0, 62)$

In the class we have 2 quizzes and 2 homework grades that are still to be determined, together with the final exam. This still gives you time to pull your grade up if you feel it's not where you want it to be. If you'd like to meet with me at any point to talk about your grade in the class to date and what you can do to improve it, please don't hesitate to send me an email.

(2) Previously in class I defined an inflexion point of a function $f(x)$ to be a place where $f''(x) = 0$. In fact, this is the incorrect definition for an inflexion point. An inflexion point is a place where concavity of f changes (either from positive to negative or negative to positive). I'm sorry for the confusion this might have caused in your mind.

2. Recap

So far in this class, we've spend lots of time motivating and developing the theory of the derivative. This derivative, of course, is the central theme of this class; it is also the star of the majority of Math 20. But what is it good for?

We have already seen two applications of the derivative. First, it can be used to approximate the value of an otherwise hard-to-evaluate function by way of tangent lines. Second, we can use the derivative to detect local extrema of a function. Today and in the next several class periods we're going to develop this second application a little more thoroughly.

3. Relative and Absolute Extrema

We have already talked about the notion of local (a.k.a. relative) maxima and minima of a function. The definition given in the book is slightly different than the one we've talked about in class. To be consistent, though, let us adopt their definition of a local maximum and minimum.

Definition. A function $f(x)$ is said to have a local (a.k.a. relative) maximum at a point c is $f(c) \geq f(x)$ for all points x near c.

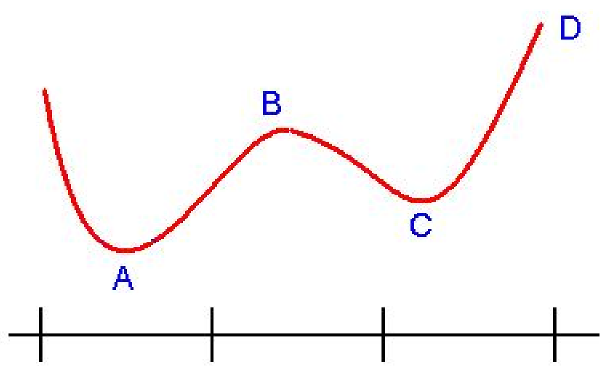
Of course, there is a corresponding definition for a local minimum of a function. The only way the new definition differs from our original definition is that we do not require a strict inequality, instead settling for an inequality. In many ways, this distinction is not important. Where can we see the difference in definitions? Originally we would not have said that the the constant function $f(x) = 1$ has any local maxima or minima, since there is no value c with $f(x) > f(x)$ for points x near c. However, since we no longer require strict inequality, we have that any point c is a local maximum of the function $f(x) = 1$ (since for any point c, $1 = f(c) \geq f(x) = 1$ for points x near c). The same reasoning gives that any point c is also a minimum of our function $f(x) = 1$.

Finding local maxima and minima of a function is important, but they only detect 'local' information. In particular, while we might have a local minimum at a point c, we do not know that c is the smallest value the function might attain. We will be interested in questions just like this in the next week. So, we adopt two new definitions.

Definition. A function $f(x)$ is said to have an absolute maximum at a point c if $f(c) \geq f(x)$ for all points x in the domain of f.

Again, there is a corresponding definition for an absolute minimum which you can guess. How are the definitions of absolute and relative extrema different? In the relative case, we only require that our point $f(c)$ beat values $f(x)$ where x is *close to* c, though in the relative case we insist that $f(c)$ beat all values $f(x)$ for x in the domain of f. Let's see the difference in action.

Example. Identify local maxima and minima and absolute maxima and minima for $f(x)$ defined by



Solution. We can see that the relative minima occur at points A and C, and that we have a relative maximum at point B. Now the function attains its maximal value at the point D, so we say f has an absolute maximum at D. Similarly, we can see that the function attains its minimal value at the point A. Hence A is the absolute minimum of f. $\square$

Notice that one might wonder in this example if the endpoints are considered relative maxima or not. In many ways, this question is really more a matter of definitions, so I'm happy to have people consider these places either as relative maxima or not as relative maxima. There are benefits to either, and I won't take off if you label these as relative maxs or not as relative maxs. I am more than happy to talk to you about this if you like, so just let me know if this troubles you.

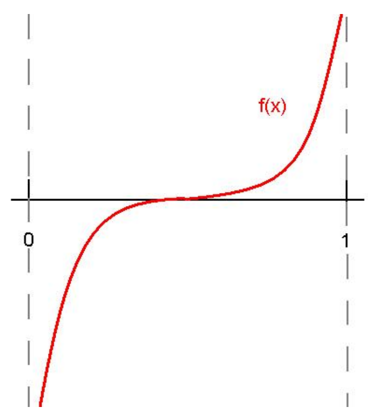
4. Extreme Value Theorem

We already have a technique for finding and identifying local maxima and minima involving the derivative. We do this by first finding where the first derivative is zero or undefined. We haven't used this terminology yet, but

Definition. A critical point of a function $f(x)$ is a place where $f'(x) = 0$ or $f'(x)$ is undefined.

After we have identified critical points, we test each critical point with either the first or second derivative test to determine where it is a local max, a local min, or neither a local max nor a local min.

How might we find and identify absolute maxima and minima? In general, this is a harder problem to solve. For one, a function might not even have a relative maximum or minimum. For instance, consider the function $f(x)$ defined on $(0, 1)$ as pictured below.

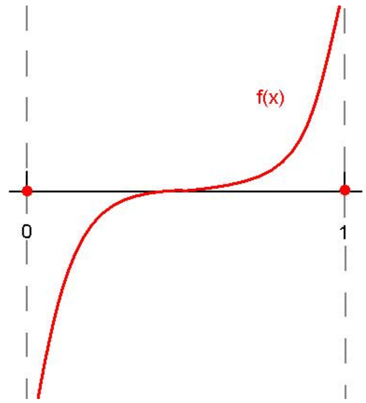


One can see that the function does not attain either a maximum value or a minimum value on the domain, since the function approaches both $+\infty$ and $-\infty$ on the domain.

So when will we know we can find extreme values (i.e., absolute maxima and minima)? In comes the

Theorem 1 (Extreme Value Theorem)**.** *If $f(x)$ is a continuous function on $[a, b]$, then $f(x)$ has an absolute maximum c and an absolute minimum d.*

Just like the intermediate value theorem, if we ever want to use the extreme value theorem to conclude a function has an absolute maximum or minimum, it is very important that the hypotheses of the extreme value theorem are satisfied. The graph above gives an example of a continuous function on the *open* interval $(0, 1)$ which does not satisfy the conclusion of the intermediate value theorem. In fact, one can also construct a function $f(x)$ defined on a closed interval which doesn't have an absolute minimum nor an absolute maximum. Such a function is shown below. Notice that it fails to be continuous on the interval $[0, 1]$!



So how can we use the extreme value theorem to find absolute maxima or minima? We'll adopt a 4 step technique:

FINDING ABSOLUTE MAXIMA AND MINIMA

1^{st} Verify that $f(x)$ is continuous on a closed interval of interest
2^{nd} Find critical points of $f(x)$ that lie in the closed interval of interest
3^{rd} Evaluate $f(x)$ at critical points and endpoints in a chart
4^{th} Pick out the absolute maximum by finding which critical point/endpoint gives the largest function value (similarly for absolute minimum)

It is *critical* if you're going to use this process that you verify the first step. If you don't first check that the function is continuous on a closed interval, this method can fail! Let's try this out.

Example. Find the absolute maximum and minimum of the function $f(x) = \dfrac{e\log(x)}{x}$ on the interval $[1, e^2]$.

Solution. We want to use the procedure above, so first we need to verify that $f(x)$ is continuous on $[1, e^2]$. Now $\log(x)$ is a nice, continuous function whenever $x > 0$, so in particular it's continuous on the interval $[1, e^2]$ we're interested in. Similarly, the denominator would only cause a problem with continuity if it vanished on the interval we're concerned with. But since 0 is not in the interval $[1, e^2]$, having that x in the denominator doesn't mess up continuity. We conclude that $f(x)$ is a continuous function on $[1, e^2]$ as desired.

For the second step, we need to find critical points of $f(x)$. This means we need to compute $f'(x)$. Now

$$\frac{d}{dx}\left[\frac{e\log(x)}{x}\right] = e\left[\frac{x\dfrac{d}{dx}\left[\log(x)\right] - \log(x)\dfrac{d}{dx}\left[x\right]}{x^2}\right] = \frac{e(1-\log(x))}{x^2}.$$

Now critical points of f are where $f'(x) = 0$ or when $f'(x)$ is undefined. Zeroes of $f'(x)$ occur when the numerator of the derivative is zero. Now $e(1 - \log(x)) = 0$ only when $1 - \log(x) = 0$, which happens only if $1 = \log(x)$, and we know that this means exactly $x = e$. $f'(x)$ is undefined only when the denominator is 0 or when the numerator is undefined, but these things only happen when $x \leq 0$. Since we're in the interval $[1, e^2]$, we don't have to worry about this. Hence, our only critical point is $x = e$.

Now we need to evaluate $f(x)$ at the critical point e and the endpoints of the interval: 1 and e^2.

x	$f(x)$
1	$f(1) = \dfrac{e\log(1)}{1} = \dfrac{e \cdot 0}{1} = 0$
e	$f(e) = \dfrac{e\log(e)}{e} = \dfrac{e \cdot 1}{e} = 1$
e^2	$f(e^2) = \dfrac{e\log(e^2)}{e^2} = \dfrac{e \cdot 2}{e} = \dfrac{2}{e}$

Since 0 is the smallest value we see for $f(x)$ in this chart, we have that 1 is the input which gives the absolute minimum of this function on $[1, e^2]$. To find the maximu, we just need to determine which is larger: 1 or $\dfrac{2}{e}$. But since $e \approx 2.7$, we see that $\dfrac{2}{e} < 1$. Hence e is the input which gives the absolute maximum of this function on $[1, e^2]$. $\square$