

COURSE NOTES - 03/07/05

1. ANNOUNCEMENTS

- (1) Our final is scheduled for Wednesday, March 16 from 8:30am to 11:30am. The final will be comprehensive.
- (2) I will post a practice final by the end of this week. It should be comparable to the exam in length and difficulty.
- (3) We'll have a review session Tuesday, March 15, starting around 6:30 pm. I'll bring dinner as I did for the first review session.

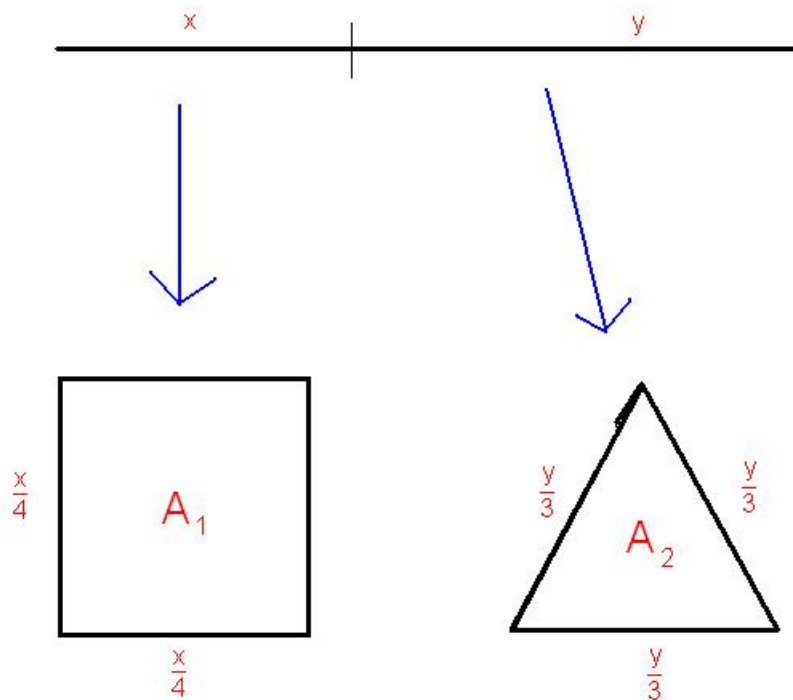
2. OPTIMIZATION PROBLEMS – ON CLOSED INTERVALS

In the next two days we're going to discuss how we apply the ideas in calculus (and in particular the topics we've discussed in the last 2 class periods) to word so-called optimization problems. These problems fall into two general categories: those which require optimization of a continuous function on a closed interval, and those that require optimization of a continuous function on an open interval. Today we'll discuss the former, though Wednesday we'll see that most of the strategy carries over to the latter as well.

An optimization problem is essentially a word problem which asks you to maximize or minimize a certain quantity. In this regard, they are incredibly similar to the sorts of problems we've been doing the last few days. Nearly all optimization problems can be solved by employing a fixed strategy. Let's try one of these problems.

Example. A piece of wire 10m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a minimum and (b) a maximum?

Solution. After I've read the problems a few times to understand what it's asking, I like to draw a picture of what's going on. In the picture I'll label certain quantities so I can keep track of what's happening. For this problem, I'll draw my piece of wire cut into two pieces (leaving two pieces of length x and y) and what these pieces of wire look like after they've been fashioned into a circle and a triangle.



We can see from the picture that the total area enclosed by both figures is

$$A = A_1 + A_2 = \left(\frac{x}{4}\right)^2 + \frac{\sqrt{3}}{4} \left(\frac{y}{3}\right)^2 = \frac{1}{16}x^2 + \frac{\sqrt{3}}{36}y^2$$

(the second term I know either from looking at the front of the book or by using the pythagorean theorem to solve for the height of the triangle). Perhaps a less obvious thing to write down is what I call a constraining equation; it's an equation that 'remembers' how the variables x and y relate to each other. In this case, the equation is given by

$$x + y = 10.$$

What I'd like to do is take my expression for area and optimize it. The problem is that currently it sits as a function of *two* variables instead of one. We don't have the technology yet to optimize a function of 2 variables. So what do we do? We'll take our constraining equation and use it to solve for y in terms of x : since $x + y = 10$, this means that $y = 10 - x$. Now I can substitute this value for y back into my expression for area to come up with an expression that makes area a function of one variable

$$A(x) = \frac{1}{16}x^2 + \frac{\sqrt{3}}{36}(10 - x)^2.$$

Now I have area as a function of the variable x . Notice that x can sit anywhere between 0 and 10, so we're now asking the following problem: find the absolute minimum (part (a)) and absolute maximum (part (b)) of the function $A(x)$ on the interval $[0, 10]$. How do we solve this problem? We use the three step technique we developed last class period.

First, we verify that $A(x)$ is a continuous function on the closed interval $[0, 10]$. Since it is a polynomial, it certainly is continuous.

Second, we find critical points for $A(x)$. Since

$$A'(x) = \frac{1}{8}x + \frac{\sqrt{3}}{18}(10 - x)(-1)$$

(I used the power rule and chain rule to compute this), we see first that the derivative is defined everywhere so that critical points are simply zeroes of the first derivative. When is $A'(x) = 0$? This

happens if and only if $\frac{1}{8}x + \frac{\sqrt{3}}{18}(10 - x)(-1) = 0$, which is equivalent to

$$\frac{1}{8}x + \frac{\sqrt{3}}{18}x = \frac{10\sqrt{3}}{18} \Leftrightarrow x = \frac{80\sqrt{3}}{18 + 8\sqrt{3}} \approx 4.35.$$

(Of course, problems you'll encounter in quizzes or tests won't require a calculator to solve, so don't be too scared by the scary numbers).

Finally, we evaluate the function at these critical points to find the maximum and minimum

x	$f(x)$
0	$\frac{10^2\sqrt{3}}{36} \approx 4.81$
4.35	$\frac{4.35^2}{16} + \frac{5.65^2\sqrt{3}}{36} \approx 2.71$
10	$\frac{10^2}{16} \approx 6.25$

From the table we see that area is maximized when $x = 10$ (that is, when all of the wire is used to make the square) and minimized when 4.35 meters is used to make the square (and so 5.65 meters is used to make a triangle). \square

In solving this problem, we've used the following steps:

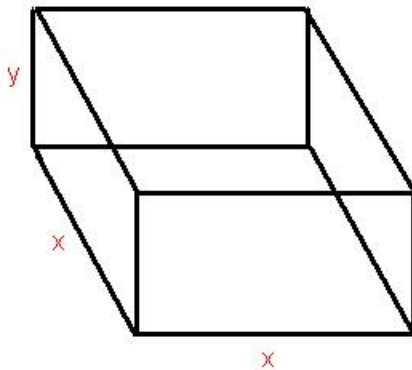
- (1) Read and re-read the problem until you understand it. In particular, make sure you know the quantity you're being asked to maximize or minimize.
- (2) Draw and label a picture which gives the relevant information.
- (3) Write equations and determine appropriate domains that describe
 - (a) the quantity you're attempting to maximize/minimize in terms of other variables which appear, and
 - (b) the constraints that your extra variables must obey.
- (4) Solve Equation (b) for one of the variables, and plug this result back into Equation (a).
- (5) Use the techniques we've discussed in the last few days to find the extrema you're interested in.

Notice that the last step in particular uses the fact that the function we're maximizing is continuous on a closed interval. On Wednesday we'll play around with some functions which are continuous on open intervals, and we'll see they require a slightly different trick to solve them completely.

Let's do one more problem.

Example. If 1200 square centimeters of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

Solution. After soaking this problem in, we draw the following sketch to give us a feel for what we're interested in:



Now we see that the volume of the box is given by the quantity

$$V = x^2 y.$$

What is the constraining equation in this instance? Since we're only given 1200 cm² of material to work with (and since our box doesn't have a top), we see that the surface area of the box must be 1200. In an equation, this says

$$1200 = x^2 + 4xy$$

(the first term is the area of the bottom piece, and the next is the surface area of the 4 side pieces). Now solving for y in terms of x , we see

$$y = \frac{1200 - x^2}{4x}.$$

Plugging this back into our expression for volume, we see

$$V(x) = x^2 \left(\frac{1200 - x^2}{4x} \right) = \frac{1200x - x^3}{4} = 300x - \frac{1}{4}x^3.$$

Notice also that x can take on values anywhere between 0 (in which case our box collapses to a line) and $\sqrt{1200}$ (in which case our box has no height). From these physical observations we see that in each of these extremal cases $V(x) = 0$.

Now since $V(x)$ is a continuous function on the closed interval $[0, \sqrt{1200}]$, we can find its absolute maximum on the interval by finding critical points and comparing the value of the function at these critical points together with the endpoints of our interval. Notice that $V'(x) = 300 - \frac{3}{4}x^2$, which

is never undefined. Hence our critical points are when $300 - \frac{3}{4}x^2 = 0$, or in other words when $300 = \frac{3}{4}x^2$. Solving for x , one finds the only critical value in the interval is $x = 20$.

Now we have already said that $V(0) = V(\sqrt{1200}) = 0$, so the maximal value of V is surely attained at $x = 20$. What is this maximal value?

$$V(20) = 6000 - 2000 = 4000.$$

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