COURSE NOTES - 03/09/05

1. Announcements

(1) Our final is scheduled for Wednesday, March 16 from 8:30am to 11:30am. The final will be comprehensive.
(2) I will post a practice final by the end of this week. It should be comparable to the exam in length and difficulty.
(3) We'll have a review session Tuesday, March 15, starting around 6:30 pm. I'll bring dinner as I did for the first review session.

2. Recap

Last class period we started investigating so-called optimization problems. In particular, we saw how one can solve optimization problems on closed intervals. We had a 5 step method for doing this which you might want to review.

3. Optimization problems – on open intervals

It often happens that you are asked to optimize a function on an open interval (or the whole real line) instead of a closed interval? How does one go about solving such a problem? Happily, much of the technique in optimizing functions on closed intervals will carry over to optimizing functions on open intervals. The main difference is that in finding absolute maxima or minima we will use a variation of the 1st derivative test instead of our techniques for finding absolute maxima and minima for continuous functions on closed intervals. This variation of the 1st derivative test is:

Theorem 1 (1st Derivative Test for absolute extrema)**.** *If $f'(a) = 0$ or undefined and $f'(x) < 0$ for $x < a$ and $f'(x) > 0$ for $x > a$, then f has an absolute minimum (in its domain) at a. If $f'(a) = 0$ or undefined and $f'(x) > 0$ for $x < a$ and $f'(x) < 0$ for $x > a$, then f has an absolute maximum (in its domain) at a.*

Notice that this is very similar to the 1st derivative test for local extrema. The difference is that in this test we require information about the sign of the derivative for *all* $x < a$ and $x > a$; in the local test, we needed to only know this information for values of x near a.

Note.

- To use the 1st derivative test for absolute extrema directly, you should be in the situation where you have exactly one critical point in your domain. If you have more than one critical point in your domain, you have to do more work.

- If you're in the situation where you have exactly one critical point (say, a) in your domain, then to test the sign of the derivative for values $x < a$ it is enough to compute the sign of the derivative for one value of $x < a$. Similarly, to test the sign of the derivative for values $x > a$ it is enough to compute the sign of the derivative for one value $x > a$. I didn't mention this in class, but I'll use it in these notes and in class on Friday.

Let's use this test to work a few examples.

Handout, #2 Find two numbers whose sum is 23 and whose product is maximum.

Solution. Normally we would start a problem like this drawing a picture. For this problem, though, there's no real need. We see we're looking for numbers x and y such that $x+y=23$ so that their product P is maximized: $P = xy$. Notice in particular that x and y have no further restriction. In particular, x could be any real number. Now since $x + y = 23$, we have $y = 23 - x$. Hence we're trying to maximize

$$P(x) = x\,(23 - x) = 23x - x^2$$

on the domain $(-\infty, \infty)$. We'll use the 1st derivative test for absolute extrema.

Now $P'(x) = 23 - 2x$, so that $x = \frac{23}{2}$ is our only critical number. *Since we're in the situation where there is only one critical number*, to find the sign of the derivative to the left of $\frac{23}{2}$ and to the right of $\frac{23}{2}$, it is enough to find the sign of the derivative at some point to the left of $\frac{23}{2}$ and at some point to the right of $\frac{23}{2}$. Now 0 is to the left of $\frac{23}{2}$, and $P'(0) = 23 > 0$. Hence $P'(x) > 0$ for values $x < \frac{23}{2}$. Similarly, since $12 > \frac{23}{2}$ and $P'(12) = 23 - 24 = -1 < 0$, we see that $P'(x) < 0$ for values $x > \frac{23}{2}$. The 1st derivative test tells us $x = \frac{23}{2}$ is a local maximum of $P(x)$, as desired.

Hence, the two numbers we're looking for are $x = \frac{23}{2}$ and $y = 23 - \frac{23}{2} = \frac{23}{2}$. □

Handout, #3 Find two positive numbers whose product is 100 and whose sum is minimum.

Solution. Just like the last problem, to solve this problem we won't be drawing any pictures. Instead, we can directly write down our constraining equation as $xy = 100$, and the quantity to be maximized is $S = x + y$. Notice that since x and y are said to be positive, the domain we're interested in is $x \in (0, \infty)$. Solving for y in the constraining equation gives $y = 100x^{-1}$, so that

$$S(x) = x + \frac{100}{x}.$$

To find the minimum of this function, our hope is to use the 1st derivative test.

Now $S'(x) = 1 - \frac{100}{x^2} = \frac{x^2-100}{x^2}$. This derivative is undefined when $x = 0$ (since then the denominator is zero), and this derivative is zero when $x^2 = 100$, which is when $x = \pm 10$. Now since $x \in (0, \infty)$, the only critical point in our domain is $x = 10$. Since we only have one critical number in our interval, to find the sign of the derivative to the left and right of $x = 10$ it is enough to sample the derivative at one point to the left and one point to the right of $x = 10$. Now $S'(1) = \frac{9^2-100}{9^2} = \frac{81-100}{81} < 0$, so that $S'(x) < 0$ for $x < 10$. Similarly, $S'(11) = \frac{11^2-100}{11^2} = \frac{121-100}{121} > 0$, so that $S'(x) > 0$ for $x > 10$. The first derivative test for absolute minima tells us that $x = 10$ is a minimum of $S(x)$.

Hence, the two numbers we're looking for are $x = 10$ and $y = 100/10 = 10$. □