DESCENT THEORY (JOE RABINOFF'S EXPOSITION)

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1. February 21

Background: EGA IV.2.

Descent theory = "notions that are local in the fpqc topology".

(Remark: we aren't assuming finite presentation, so flat isn't an open condition.)

What is the fpqc topology?

Fix a scheme S. Consider $Sch/S = \{\text{schemes}/S\}$. A set $\{f_i : U_i \to X\}$ is an *fpqc cover* provided that

(i) each f_i is flat.
(ii) for every quasicompact open $U \subset X$, there exist finitely many quasicompact opens $V_i \subset U_i$ such that $U = \cup_i f_i(V_i)$ (as sets). This implies: $\coprod V_i \to U$ is faithfully flat (= flat and surjective), and if U is affine, then $\coprod V_i \to U$ is fpqc.

Translation: this topology is generated by the notion of Zariski-open sets and fpqc maps.

Note:

(1) A Zariski cover is an fpqc cover.
(2) A surjective fpqc morphism $U \to X$ is an fpqc cover.
(3) "In some sense, the covers are generated by this".

Fact: A presheaf $\mathcal{F}$ on Sch/S is an fpqc sheaf if and only if:

(i) for all Zariski covers $\{U_i \to X\}$,
$$\mathcal{F}(X) \to \prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \cap U_j)$$
is exact.
(ii) for all fpqc morhpisms $X' \to X$
$$\mathcal{F}(X) \to \mathcal{F}(X') \rightrightarrows \mathcal{F}(X' \times_X X')$$
is exact.

Date: Tuesday, February 21, 2006.

Fact 2: A property is fpqc-local on the base if it is local with respect to both kinds of covers. He'll discuss this next week.

Why do we care?

(1) fpqc $\supset$ fppf $\supset$ (big) smooth $\supset$ (big) etale $\supset$ finite etale "$\supset$ Zariski."

(2) Lots of notions are fpqc-local. This is handy. If you want to check if $X \to Y$ over some field k has some property, you can base change to the algebraic closure.

Types of descent.

The first two of the below are the core of descent theory.

(1) *Descent of morphisms:* Let S be a scheme, $p : S' \to S$ is a fpqc morphism. Let X be an "object" (by which mean a scheme, or a quasicoherent sheaf, or a locally free coherent sheaf, etc.) over S. Let p^*X be its pullback to S'. Let Y be another "object".

Question: Given a morphism $p^*X \xrightarrow{\phi'} p^*Y$, when is there $\phi : X \to Y$ such that $\phi' = p^*\phi$.

We'll do this today, and it will imply that the functor of points is an fpqc-sheaf.

(2) *"Effective descent", or descent of objects.* (This should be done at the same time as (1).)

In the situation in (1), given X' on S', when is there an X on S such that $X' = p^*X$.

(One way of thinking of (1) and (2) is as an equivalence of categories, from things over S, to "things over S' with descent data".)

(3) *Descent of properties of morphisms.* When is a property of morphisms fpqc-local (on the target).

(4) *Quotients.* If you have a group scheme acting a scheme. Sometimes you can show that the quotient exists fpqc-locally. Then you descend. (We're not going to discuss this in these notes.)

Remark: here is an interesting fpqc cover: Let C be a curve over a field k, and p a point. Then $(C - p) \coprod \operatorname{Spec} \mathcal{O}_{C,p} \to C$ is an fpqc cover.

(1) Descent of morphisms.

Suppose $p : S' \to S$. Let $\mathcal{F}, \mathcal{G}$ be quasicoherent sheaves on S, and let $\phi' : p^*\mathcal{F} \to p^*\mathcal{G}$ be a homomorphism. When is $\phi' = p^*\phi$, $\phi : \mathcal{F} \to \mathcal{G}$?

There is an obvious necessary condition. Let $S'' = S' \times_S S$, $q_1, q_2 : S'' \to S'$ the two projections. Let $q = p \circ q_1 = p \circ q_2$.

$$\begin{tikzcd} & S'' \arrow[dl, "q_1"'] \arrow[dr, "q_2"] \arrow[dd, "q"] & \\ {} \arrow[dr, "p"'] & & {} \arrow[dl, "p"] \\ & {} & \end{tikzcd}$$

For any $\phi : \mathcal{F} \to \mathcal{G}$,

$$q_1^* p^* \phi = q^* = q_2^* p^* \phi$$

. So this is a *necessary* condition for phi' to descend, i.e. $q_1^* \phi' = q_2^* \phi'$.

Then this is a *sufficient* condition as well; and if ϕ' descends, then it descends uniquely. Precisely:

Theorem. The sequence of sets

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \xrightarrow{p^*} \mathrm{Hom}(p^*\mathcal{F}, p^*\mathcal{G}) \overset{q_1^*, q_2^*}{\Longrightarrow} \mathrm{Hom}(q^*\mathcal{F}, q^*\mathcal{G})$$

is exact.

Proof. This is local on S, so we may assume that $S = \operatorname{Spec} R$. So S' is quasicompact, so $S' = \cup_{\text{finite}} S_i$ where S_i is affine. Let $\overline{S} = \coprod S_i$ — note that this is affine. Then $\overline{S} \to S' \to S$. Then a boring diagram chase shows that it suffices to show this with S' replaced by $\overline{S}$.

So we assume $S' = \operatorname{Spec} R'$, and $R \to R'$ faithfully flat.

Suppose we knew that for all R-modules M,. the sequence

$$M \longrightarrow R' \otimes_R M \Longrightarrow R' \otimes_R R' \otimes_R M$$

is exact. (This is the only place that faithful flatness will be used!)

If $\mathcal{F} = \tilde{M}$, $\mathcal{G} = \tilde{N}$. Then we want to find some ϕ in the following diagram.

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d, "\phi?"] & M \otimes_R R' \arrow[r, Rightarrow] \arrow[d, "\phi'"] & M \otimes_R R' \otimes_R R' \arrow[d, Rightarrow, "\text{these arrows agree by hypothesis}"] \\ N \arrow[r, hook] & N \otimes_R R' \arrow[r, Rightarrow] & N \otimes_R R' \otimes_R R' \end{tikzcd}$$

Diagram chase, get ϕ.

So we now deal with that useful fact.

Lemma. Let $R \to R'$ be a faithfully flat extension of rings. Then for all R-modules M,

$$0 \to M \to M \otimes_R R' \to M \otimes_R R' \otimes_R R'$$

is exact.

Proof. This is exact iff it is so after faithfully flat base change $\otimes_R R'$:

$$0 \to M \otimes_R R' \to M \otimes_R R' \otimes_R R' \to M \otimes_R R' \otimes_R R' \otimes_R R'$$

is exact.

Idea: $R \to R'$ is our morphism, and after this base change, it has a section $R' \otimes_R R' \to R'$ — the multiplication map.

Then (according to Neron models) it is now elementary to check that it is exact now; Joe refers to Kirsten. $\square$

Thus we have proved descent for quasicoherent sheaves.

Corollary: Let $\mathcal{F}$ be a quasicoherent sheaf on S. Define a presheaf $\hat{\mathcal{F}} : \mathrm{Sch}/S \to \text{Sets}$ by $\hat{\mathcal{F}}(f : X \to S) = \Gamma(X, f^*\mathcal{F})$. Then $\hat{\mathcal{F}}$ is a sheaf.

Proof. It is enough to check on Zariski covers and fpqc morphisms. Zariski covers is clear, so we consider an fpqc morphism $S' \to S$. Take $\mathcal{G} = \mathcal{F}$, $\mathcal{F} = \mathcal{O}_S$ from before.

$$\begin{tikzcd} \mathrm{Hom}(\mathcal{O}_S, \mathcal{F}) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(\mathcal{O}_{S'}, p^*\mathcal{F}) \arrow[r, Rightarrow] \arrow[d, equal] & \mathrm{Hom}(\mathcal{O}_{S'}, q^*\mathcal{F}) \arrow[d, equal] \\ \hat{\mathcal{F}}(S) \arrow[r] & \hat{\mathcal{F}}(S') \arrow[r, Rightarrow] & \hat{\mathcal{F}}(S'') \end{tikzcd}$$

Theorem. Let $p : S' \to S$ be fpqc, let $X', Y'/S$ be schemes $X' = X \times_S S'$, $Y' = Y \times_S S'$, then

$$\mathrm{Mor}_S(X, Y) \xrightarrow{p^*} \mathrm{Mor}_{S'}(X', Y') \Longrightarrow \mathrm{Mor}_{S''}(X'', Y'')$$

is exact.

Proof idea: First S affine, then to S'' affine by the same trick. Then asume that Y is affine. Then that sequence above looks like

$$\mathrm{Hom}_R(B, \Gamma(X, \mathcal{O}_X) \to \cdots .$$

Wickedly important corollary. If X/S is a scheme, and $\underline{X}(T) := \mathrm{Mor}_S(T, X)$. Then $\underline{X} : \mathrm{Sch}/S \to \text{Sets}$ is an fpqc sheaf.

Proof. It suffices to check on Zariski covers and fpqc morphisms. The Zariski is easy, and we've done it, so we deal with an fpqc map $T' \to T$.

Then:

$$\begin{tikzcd} \mathrm{Mor}(T, X) \arrow[r] \arrow[d, equal] & \mathrm{Mor}(T', X) \arrow[r, Rightarrow] \arrow[d, equal] & \mathrm{Mor}_S(T'', X) \arrow[d, equal] \\ \mathrm{Mor}_T(T, T \times_S X) \arrow[r] & \mathrm{Mor}_{T'}(T', T' \times_S X) \arrow[r, Rightarrow] & \mathrm{Mor}_{T''}(T'', T'' \times_S X) \end{tikzcd}$$

2. February 28

We are investigating what notions are "local in the fpqc topology".

We discussed: (1) Descent of morphisms.

This is like a single overlap condition. The following will be like a cocyle condition.

(2) Effective descent (=descent of objects).

Given $p : S' \to S$ that is fpqc.

Given an object $\mathcal{F}$ on S', when does there exist $\mathcal{F}$ on S such that $\mathcal{F}' = p^*\mathcal{F}$?

This will be a sheafy condition.

A special case of this is Galois descent. We'll see an example of that shortly.

(3) Descent of properties of morphisms = when is a property of morphisms fpqc-local. The answer is almost always (unless the property is "projective").

(4) "Quotients". We won't be talking about that.

We'll now discuss **(2)**. In order to do that, we'll discuss (1) again.

Fix $p : S' \to S$ fpqc morphism. Define $S'' = S' \times_S S'$, $S''' = S' \times_S S' \times_S S'$, $p_1, p_2 : S'' \to S'$, $p_{ij} : S''' \to S''$ the obvious projections.

$$\begin{tikzcd}
 & S''' \arrow[dl, "p_{12}"'] \arrow[d, "p_{13}"] \arrow[dr, "p_{23}"] & \\
S'' \arrow[d, "p_1"] \arrow[dr, "p_1"] & S'' \arrow[dl, "p_2"] \arrow[dr, "p_1"] & S'' \arrow[dl, "p_2"] \arrow[d, "p_2"] \arrow[ddl, "q"] \\
S' \arrow[dr, "p"] & S' \arrow[d, "p"] & S' \arrow[dl, "p"] \\
 & S &
\end{tikzcd}$$

Theorem. For sheaves $\mathcal{F}, \mathcal{G}$ on S, the following sequence is exact:

$$\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{G}) \xrightarrow{p^*} \mathrm{Hom}_{\mathcal{O}_{S'}}(\mathcal{F}', \mathcal{G}') \rightrightarrows^{p_1^*, p_2^*} \mathrm{Hom}_{\mathcal{O}_{S''}}(\mathcal{F}'', \mathcal{G}'')$$

Definition: the category of *sheaves on S' with covering data* is the following.

Objects are pairs $(\mathcal{F}', \phi)$, where $\mathcal{F}'$ is a quasicoherent sheaf on S, an isomorphism $\phi : p_1^*\mathcal{F}' \to p_2^*\mathcal{F}'$.

Morphisms are $f : \mathcal{F}' \to \mathcal{G}'$ such that the following diagram commutes.

$$\begin{tikzcd}
p_1^*\mathcal{F}' \arrow[r, "p_1^* f"] \arrow[d, "\phi", "\sim"'] & p_2^*\mathcal{G} \arrow[d, "\psi"', "\sim"] \\
p_1^*\mathcal{F}' \arrow[r, "p_2^* f"] & p_2^*\mathcal{G}.
\end{tikzcd}$$

The theorem above can now be written as:

Theorem. The functor $\mathcal{F} \mapsto (p^*\mathcal{F}, \text{canonical})$ is fully faithfully from $Qcs(S) \to$ sheaves on S' with covering data.

Idea: As stated earlier, this will be like gluing sheaves.

First note that if $\mathcal{F}' = p^*\mathcal{F}$, $\phi : p_1^*\mathcal{F}' \to p_2^*\mathcal{F}'$ canonical, we get a commuting hexagon

$$\begin{tikzcd}
& p_{12}^*p_2^*\mathcal{F}' \arrow[dr, "="] & \\
p_{12}^*p_1^*\mathcal{F}' \arrow[ur, "p_{12}^*\phi"'] & & p_{23}^*p_1^*\mathcal{F}' \arrow[d, "p_{13}^*\phi"] \\
p_{13}^*p_1^*\mathcal{F}' \arrow[u, "="] \arrow[dr, "p_{13}^*\phi"'] & & p_{23}^*p_2^*\mathcal{F}' \arrow[dl, "="] \\
& p_{13}^*p_2^*\mathcal{F}' &
\end{tikzcd}$$

which basically says that all six ways of pulling back all the way to S''' are the same.

The slick way of saying this in terms of cocycles is: $p_{23}^*\phi \circ p_{12}^*\phi = p_{13}^*\phi$.

Definition: The *category of quasicoherent sheaves on* S' *with descent data* is the full subcategory of the category of quasicoherent sheaves on S' with covering data such that this hexagon cocyle condition holds.

Theorem. $\mathcal{F} \mapsto (p^*\mathcal{F}, \text{canonical})$ is an equivalence from $Qcs(S) \to$ category of quasicoherent sheaves with descent data.

Here is lemma number 1, which is where fpqc comes up.

Lemma. If $p : S' \to S$ admits a section $s : S' \to S$, then the $\mathcal{F}' \mapsto s^*\mathcal{F}'$ is the inverse functor.

The proof is formal and a bit boring.

Then the idea in general is that if you don't have a section, you base change so that you do.

Let's now prove the theorem. We have two natural maps

$$\mathcal{F}' \to p_{1*}p_1^*\mathcal{F}',$$

$$\mathcal{F}' \longrightarrow p_{2*}p_2^*\mathcal{F}' \xrightarrow{\phi^{-1}} p_{2*}p_1^*\mathcal{F}' .$$

Pushing these forward by p_*, we get $\alpha : p_*\mathcal{F}' \to q_*p_1^*\mathcal{F}'$, $\beta : p_*\mathcal{F}' \to q_*p_1^*\mathcal{F}'$. Let $\mathcal{F} = \ker(\alpha - \beta)$. Claim (without proof): $(p^*\mathcal{F}, \text{canonical}) \cong (\mathcal{F}', \phi)$.

But why is this the right way to determine $\mathcal{F}$? The motivation is what we did last time, when we said that

$$M \to M \otimes_R R' \rightrightarrows M \otimes_R R' \otimes_R R'.$$

So if we didn't know M, but we knew the next two things, then we'd hope to recover M as a kernel.

Descending schemes is trickier, because the cocycle condition (although necessary) is no longer sufficient. (One vague reason: you can no longer reduce to the affine case.)

He's now giving reasons why we need more conditions. Suppose $X' = X \times_S S'$. Say $p : S' \to S$ finite implies $\pi : X' \to X$ finite. Recall that $\phi : p_1^*X' \to p_2^*X'$ is the canonical isomorphism. That means that $\pi^{-1}U$ is affine when U is affine.

X' can be covered with affine open sets such that $\phi(p_1^{-1}\pi^{-1}(U)) = p_2^{-1}(\pi^{-1}(U))$ holds.

Theorem. (a) The functor $X \mapsto X \times_S S' = p * X$, $Sch/S \to S'$ with covering data is fully faithful. (We did this last time. This implied that the functor of points is a fpqc sheaf.)

(b) (This is just one example of a criterion for descent to be effective. There are many more. We say that a descent datum is effective if it actually descends.) Suppose S, S' are affine, and (X', ϕ) is a descent datum of schemes on S' that can be covered with (quasi-)affine U_i *stable under* ϕ, i.e. $\phi(p_1^{-1}(U_i)) = p_2^{-1}(U_i)$, Then (X', ϕ) is *effective*, i.e. $(X', \phi) = p^*X$ for some X.

We'll now do the example of **Galois descent**.

Definition. A *Galois covering* $p : S' \to S$ is an fpqc map such that there is a finite group Γ, acting on S'/S, such that $\Gamma \times S' \to S' \times_S S'$ $((\sigma, s) \mapsto (\sigma, s))$ is an isomorphism. Here $\Gamma \times S' = \coprod_{\sigma \in \Gamma} S'$.

Example: $S = \operatorname{Spec} k$, k'/k Galois. $S' = \operatorname{Spec} k'$, $\Gamma = Gal(k'/k)$. Note that $k' \otimes_k k' = \prod_{g \in \Gamma} k'$.

Let X' be an S'-scheme with a Γ-action such that

$$\begin{tikzcd} \Gamma \times X' \arrow[r] \arrow[d] & X' \arrow[d] \\ \Gamma \times S' \arrow[r] & S' \end{tikzcd}$$

commutes.

We claim that this corresponds naturally to a descent datum.

Here is another way to think about a covering datum: X' with an isomorphism $X' \times_S S' \to S' \times_S X'$ over $S\ times_S S'$. (The left side is $p_1^* X'$, the right side is $p_2^* X'$.) This is equivalent to having X'' equipped with $q_1, q_2 : X'' \to X'$ such that

$$\begin{tikzcd} X'' \arrow[r, "q_i"] \arrow[d] & X' \arrow[d] \\ S'' \arrow[r, "p_i"] & S' \end{tikzcd}$$

is Cartesian for $i = 1, 2$.

Similarly, descent datum is the same as multi-Cartesian diagrams

$$\begin{tikzcd} X''' \arrow[r, "q_{ij}"] \arrow[d] & X'' \arrow[d] \\ S''' \arrow[r, "p_{ij}"] & S'' \end{tikzcd}$$

are Cartesian for all ij, and

$$\begin{tikzcd} X'' \arrow[r, "q_i"] \arrow[d] & X' \arrow[d] \\ S'' \arrow[r, "p_i"] & S' \end{tikzcd}$$

are Cartesian for all i. $X'' \cong \Gamma \times X'$ implies $X''' \cong \Gamma \times \Gamma \times X'$. Define

$$\Gamma \times \Gamma \times X' \Rrightarrow \Gamma \times X' \rightrightarrows X' .$$

Were the three maps on the left are defined by (respectively)

$$(\sigma, \tau, x) \mapsto \begin{cases} (\sigma, \tau x) \\ (\sigma\tau, x) \\ (\tau, x) \end{cases}$$

and the two maps on the right are given by

$$(\sigma, x) \mapsto \begin{cases} \sigma x \\ x \end{cases}$$

Then this is a descent datum (under this translation).

Theorem. Assume S, S' affines, X' separated. Then the above descent datum is effective if and only if for all orbits in X' are contained in some affine.

This follows from similar things to what we've done before.

3. MARCH 7

Definition. A property P of morphisms of S-schemes if *fpqc-local on the base/target* provided that

(1) If $f : X \to Y$ has P and $\{Y_i \to Y\}$ is an fpqc cover then $X \to Y$ has P.
(2) If there exists a cover $\{Y_i \to Y\}$ such that $f_i : X \to Y$ has P for all i implies f has P_i.

Lemma 2. Let P be a property of morphisms of S-schemes that is:

(1) Stable under base change
(2) Zariski-local on target
(3)

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d, "\text{fpqc}"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

f has P if and only if f' has P.

(One motivation for fpqc topology: you can check things over $\mathbb{F}_p$ by checking over the algebraic closure.)

Theorem (EGA IV.2.2). The following 18 properties of morphisms are fpqc-local on the base: flat, quasiseparated, locally of finite type, locally of finite presentation, separated, finite type, finite presentation, proper (harder one, uses something fancy), isomorphism, monomorphism, open immersion, closed immersion, affine, quasi-affine, finite, quasifinite, surjective.

Let's check flatness. Reduce to the affine case, as this notion is local on both the source and target.

$$\begin{tikzcd} A \otimes_B B' & B' \arrow[l, "\text{flat}"'] \\ A \arrow[u] & B \arrow[l, "\text{flat?}"] \arrow[u, "\text{fpqc}"'] \end{tikzcd}$$

The strategy is to get from B to A via the scenic route.

Suppose we have an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of B-modules. We want to check if this remains exact upon tensoring with A. Apply $B' \otimes_B$:

$$0 \to B' \otimes_B M' \to B' \otimes_B M \to B' \otimes_B M'' \to 0$$

so $0 \to (A \otimes_B B') \otimes_B M' \to \cdots$ is exact

Rearrange: $0 \to B \otimes_B (A \otimes_B M') \cdots$

Then by faithful flatness, "erase" the B.

Quotients are in SGA III vol. 1 expose 5.

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