AN OVERVIEW OF FUNCTIONS

In class today we talked about all things related to real-valued functions.

1. Introduction

Definition 1.1. A real-valued function $f : D \to \mathbb{R}$ is a rule which assigns to each input in the set D an output in the set of real numbers $\mathbb{R}$.

There are a wealth of basic real-valued functions that you have seen previously in your life. These include:

- polynomials (like $x^2 + 1$ or $x^{17} + x^2 + x$)
- rational functions (quotients of polynomials, such as $\dfrac{x^2+1}{x^{17}+x^2+x}$ or $7x + x^{-2}$)
- algebraic functions (functions that can be written by adding, multiplying, dividing, or extracting roots of polynomials, such as $\sqrt{\dfrac{x^2+1}{x^{17}+x^2+x}}$)
- trigonometric functions (these are $\sin(x)$, $\cos(x)$, $\tan(x)$, $\csc(x)$, $\sec(x)$, or $\cot(x)$)
- exponentials functions (such as e^x or 2^x)
- logarithmic functions (such as $\ln(x)$ or $\log_2(x)$)
- the absolute value function (written $|x|$)

2. Combining functions

Although there are lots of 'basic' functions we could talk about, we can create really interesting functions by combining those functions found in our library above. There are a few ways to combine functions. For instance, we can take two old functions and define a new function by adding them together. The sum of two functions f and g is defined by the rule

$$(f + g)(x) = f(x) + g(x).$$

Similarly we can take the product of two functions by multiplying their outputs

$$(fg)(x) = f(x)g(x).$$

A far more subtle way to combine functions is function composition, where we evaluate one function 'at' another function. The composition of f and g is written as $f \circ g$, and is defined by the rule

$$(f \circ g)(x) = f(g(x)).$$

Example. Let $f(x) = x^2 + 1$, $g(x) = \sin(x)$, and $h(x) = e^x$. What are $f + h$, fh, and $f \circ h$? What is $h \circ f \circ g$?

Solution. The function $f+h$ is defined by the rule $(f+h)(x) = f(x)+h(x)$, and of course $f(x)+h(x) = x^2+1+e^x$.

Similarly fg is defined by the rule $(fg)(x) = f(x)g(x)$, so that $(fg)(x) = (x^2 + 1)e^x$.

The function $f \circ h$ is given as

$$(f \circ h)(x) = f(h(x)) = f(e^x) = (e^x)^2 + 1 = e^{2x} + 1.$$

Finally, the function $h \circ f \circ g$ is

$$\begin{aligned}(h \circ f \circ g)(x) &= h(f(g(x))) = h(f(\sin(x))) \\ &= h((\sin(x))^2 + 1) = h(\sin^2(x) + 1) \\ &= e^{\sin^2(x)+1}.\end{aligned}$$

□

It is also critical in this class that you are able to take a given expression and identify it as a composition of functions. When doing this, it's good to work from the outside in.

Example. Write $\sin(2^{\tan(x+1)})$ as a composition of functions.

Solution. The answer is $(\sin(x)) \circ (2^x) \circ (\tan(x)) \circ (x + 1)$.

To see this, I start by asking 'What is the last thing I do when evaluating this function?' In this case, the last thing to do is evalue the sin of some number, and therefore $\sin(x)$ will be the last function I compose with (i.e., it goes on the far left). Then I forget about the sin in my expression (so that it becomes $2^{\tan(x+1)}$ and ask 'What is the last thing I do when evaluating this function?' The answer is now that I raise 2 to a paricular power, so that the function 2^x is the second-to-last function in my composition (i.e., it goes to the right of $\sin(x)$). Just as before, I now forget about the 2 and look only at the remaining function $\tan(x + 1)$, and I ask 'What's the last thing I do when evaluating this function?' Here the answer is that I evaluate the tangent of a number, so that $\tan(x)$ belongs in my composition. This leaves me with only $x + 1$, which goes on the far right of my expression.

Note: This is easier to see me do in person, so if you want to see this again just ask! □

3. Function inverses

Now that we know how to combine functions, it is useful to know how to 'uncombine' them. That is, if I've put f and g together to form a new function, how can I recover either f or g from the new function?

If I want to recover f from $f + g$, we simply subtract g from $f + g$, just like you'd expect. Similarly if we want to recover f from fg, we just divide by g. But how do I recover f from $f \circ g$? This is a far subtler question, and leads us to the concept of the inverse of a function.

Definition 3.1. For a function f and a value a in the range of f, $f^{-1}(a)$ is the set of solutions to $f(x) = a$.

In other words, $f^{-1}(a)$ is the collection of all those inputs whose output is a.

Example. Let $f(x) = x^2$. What is $f^{-1}(4)$?

Solution. We know that $f^{-1}(4)$ is the set of all solutions to the equation $f(x) = 4$. Since $f(x) = x^2$, this means we are looking for solutions to $x^2 = 4$. Of course there are just the values $x = \pm 2$, so that $f^{-1}(4) = \{-2, 2\}$. $\square$

This example is an important one to keep in mind, because it shows us that the inverse of a function need not be a function, since for some a in the range of f it may be the case that $f^{-1}(a)$ is a set with more than one element.

Frequently we will be interested finding the inverse of a function for an arbitrary input. In order to do this, first write $y = f(x)$. Now switch the place of y and x in the expression, and solve this new expression for y. This will give you the inverse of your function.

Example. Let $f(x) = 2x + 1$. What is $f^{-1}(x)$?

Solution. We are given $y = f(x) = 2x + 1$. To solve for $f^{-1}(x)$ we are first supposed to switch y and x in this equation, so we have $x = 2y + 1$. Now we solve for y in this expression, and the result gives us $f^{-1}(x)$. Using basic algebra techniques provides

$$f^{-1}(x) = \frac{x-1}{2}.$$

$\square$

Example. Let $f(x) = e^{2x}$. What is $f^{-1}(x)$?

Solution. Again, since our equation is $y = e^{2x}$ we are meant to solve for y in the expression $x = e^{2y}$. To do this we take natural logs of both sides. On the left this gives $\ln(x)$, and on the right this gives $\ln(e^{2y})$. But $\ln(e^{2y}) = 2y$, and so we are left with $\ln(x) = 2y$. Hence we have

$$f^{-1}(x) = \frac{\ln(x)}{2}.$$

$\square$

The nice property of the inverse of a function is that for a function f, if its inverse is also a function then $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$ (when $f^{-1}(x)$ isn't a function, we can only say $(f \circ f^{-1})(x) = x$). Because this is true, we can use inverses to 'undo' function composition: if we want to recover g from $f \circ g$, we simply compose with f^{-1}:

$$\left(f^{-1} \circ (f \circ g)\right)(x) = \left((f^{-1} \circ f) \circ g\right)(x) = (x \circ g)(x) = g(x).$$

Similarly to recover f we precompose with g^{-1}: $(f \circ g \circ g^{-1})(x) = f(x)$.

Of all the inverses we talk about, the two that will be important to keep in mind are (in order of importance) that $\ln(x)$ and e^x are inverse functions (and similarly $\log_a(x)$ and a^x are inverses for any $a > 0$); and that $\sin(x)$ and $\arcsin(x)$ (or $\cos(x)$ and $\arccos(x)$, or $\tan(x)$ and $\arctan(x)$, etc) are inverse functions.

4. Graphs of functions

You'll be expected to know the graphs of familiar functions, including (but not limited to) the graphs of a line, parabola, cubic, sine, cosine, tangent, exponential function, and logarithmic function. Given the graph of a function $f(x)$ you will also be expected to be able to sketch the graph of $f(x+1)$, $f(2x)$, $2f(x)$, etc.

Given a graph, you can determine if it is the graph of a function using the vertical line test as follows. If you are able to find a vertical line which hits the given graph in more than one spot, than the graph fails the vertical line test and is not a function (why? because there is an input which yields two outputs). If there is no vertical line which hits the graph in more than one spot, than you have a bona fide function.