AN INTRODUCTION TO THE TANGENT PROBLEM

1. Recap

Last class period we blitzed through the highlights of algebra, reviewing most of the critical tools we'll need to carve through the thicket that is calculus. These included

- functions (defining the notion of a real-valued function, building a 'library' of basic functions);
- ways of combining functions (addition, multiplication, and composition);
- the inverse of a function (the definition of the inverse, how one computes the inverse of a function, and inverses of some of our favorite functions); and
- graphs (the graphs of some basic functions in our library, how to tell if a graph represents a function via the vertical line test).

2. Loose Ends: A Fantastic Function Finale

2.1. **Properties of logarithms.**

First, there are a handful or rules about logarithms which you will want to have ready at your fingertips. These rules correspond to some rules about exponentials you probably feel more comfortable with, so I'll include the corresponding exponential properties along with the logarithmic ones.

$$\begin{array}{lcl} \log_a(xy) = \log_a x + \log_a y & \leftrightarrow & a^x a^y = a^{x+y} \\ \log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y & \leftrightarrow & \dfrac{a^x}{a^y} = a^{x-y} \\ \log_a(x^n) = n\log_a(x) & \leftrightarrow & a^{x\cdot n} = (a^x)^n \\ \log_a(1) = 0 & \leftrightarrow & a^0 = 1 \end{array}$$

2.2. **The inverse of a composition of functions.**

A second (and unrelated) problem is the following: given the composition of functions $f \circ g$, what is $(f \circ g)^{-1}$? One can verify that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ by checking that

$$(f \circ g) \circ (g^{-1} \circ f^{-1}) = (g^{-1} \circ f^{-1}) \circ (f \circ g) = x.$$

We'll check one of these equalities now, but let me first remind you that for any function $h(x)$ we have $h(x) \circ x = h(x)$. Indeed the left hand side just means 'evaluate the function $h(x)$ at x,' and when we do this we're just left with $h(x)$ (the right hand side). Ok, with that out of the way, let's verify $(f \circ g) \circ (g^{-1} \circ f^{-1}) = x$:

$$\begin{aligned} (f \circ g) \circ (g^{-1} \circ f^{-1}) &= f \circ (g \circ g^{-1}) \circ f^{-1} \\ &= f \circ x \circ f^{-1} \\ &= f \circ f^{-1} \\ &= x \end{aligned}$$

Example. What is the inverse of $f(x) = e^{\tan(x)}$?

Solution. We notice that $f(x) = e^x \circ \tan(x)$. Further we remember that the inverse of e^x is $\ln(x)$ and that $\arctan(x)$ is the inverse of $\tan(x)$. Hence our rule above says that

$$f^{-1}(x) = \arctan(x) \circ \ln(x) = \arctan(\ln(x)).$$

$\square$

2.3. **Graphing the inverse of a function.** While we're on the topic of inverses, given the graph of a function $f(x)$ it is useful to know how to graph its inverse. Thankfully there is a nice procedure for doing exactly this. First, sketch the line $y = x$ in your graph of $f(x)$. Then reflect the given graph of $f(x)$ across the line $y = x$. The reflection you've drawn is the graph of $f^{-1}(x)$. Notice that sketching the graph of $y = x$ can be tricky if the axes are wonky.

Example. Suppose $f(x)$ has the graph given below. What is the graph of $f^{-1}(x)$?

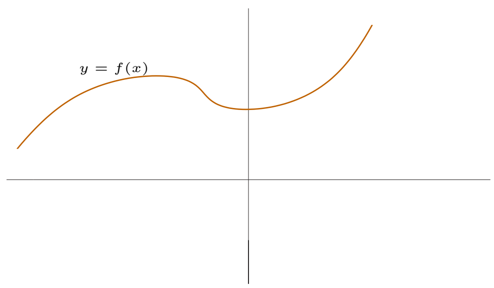
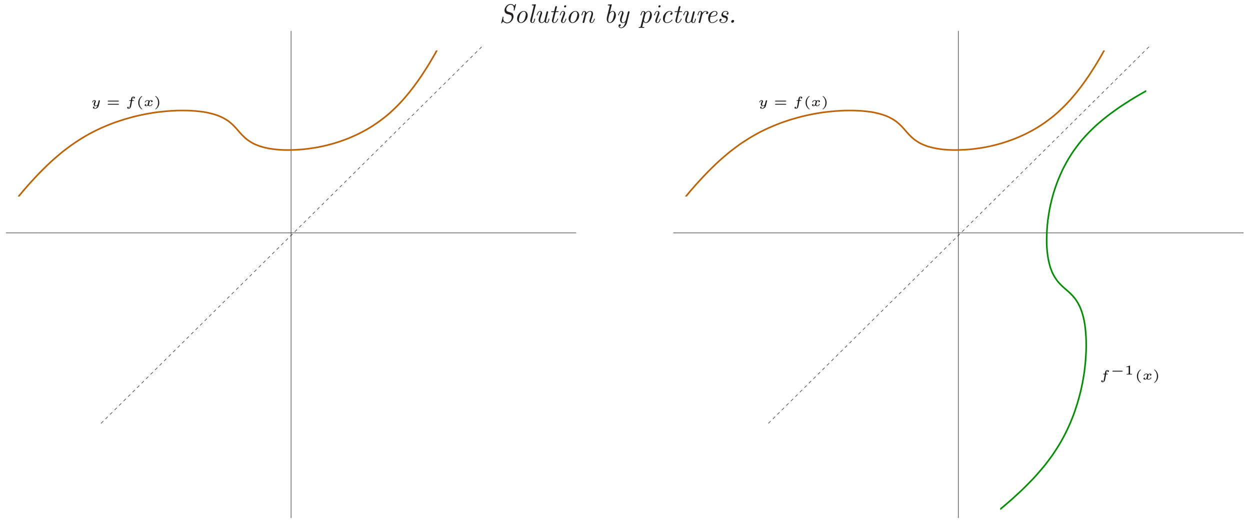


FIGURE 1. The function $f(x)$

Solution by pictures.



Including the line $y = x$ The reflection across $y = x$

FIGURE 2. Graphing $f^{-1}(x)$

$\square$

2.4. **Writing the equation of a line.** Calculus centers around writing the equation of the line tangent to a given function at a given point. Therefore, it's not unreasonable that we'll need to feel pretty comfortable writing equations of lines. In order to write the equation of a line, we need to have some information. In particular, we can write the equation of a line if we have any of the following information

- the slope of the line (call it m) and the y-intercept of the line (call it b)
- the slope of the line (call it m) and an arbitrary point of the line (say (x_0, y_0))
- two points on the line (call them (x_0, y_0) and (x_1, y_1)).

For instance, if m is the slope of the line and b is the y-intercept, then the equation of the line is given by the so-called *slope-intercept form*:

$$y = mx + b.$$

If we are given the slope m and an arbitrary point (x_0, y_0) then the equation of the line is given by the *point-slope form*:

$$y - y_0 = m(x - x_0).$$

Finally, if we have two points on the line (x_0, y_0) and (x_1, y_1), then we can write the slope as

$$m = \frac{y_0 - y_1}{x_0 - x_1} = \frac{y_1 - y_0}{x_1 - x_0}.$$

Once we've done this, we can appeal to the point-slope form to write the equation of the line.

3. An Introduction to Calculus

We've gone through all the algebra fundamentals, and now we're ready to start thinking about calculus! It helps to know what this means exactly, so allow me to introduce you to the basic problem in differential calculus: the tangent problem.

The tangent problem asks the following question. Given a function $f(x)$ and a point $P = (x_0, f(x_0))$ on the graph of $f(x)$, what is the equation of the line tangent to $f(x)$ at $(x_0, f(x_0))$? Of course to answer this question,

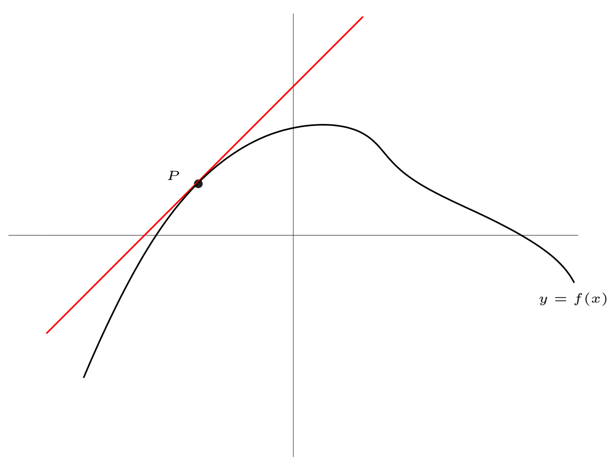


Figure 3. The function $f(x)$ and the tangent at $P = (x_0, f(x_0))$

we know from our investigation of lines that we need to know either 2 points on the tangent line or the slope of the tangent line together with a point on the line. But happily we already have a point on the graph (namely $(x_0, f(x_0))$), so we only need to know the slope of the tangent line. How will we do this?

The big idea is to consider secant lines. Specifically, plop a point $Q = (x_1, f(x_1))$ down on the graph of $f(x)$ and write the equation of the line through P and Q (you can do this since 2 points determine a line, just like we saw before). This is the so-called secant line through P and Q.

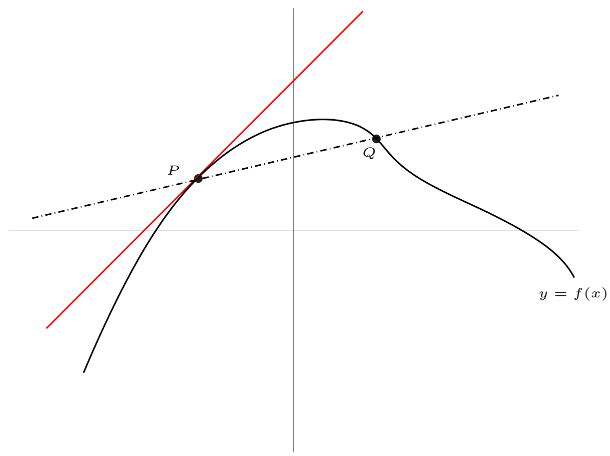


FIGURE 4. The secant line through P and Q

Now the magic: if we let Q approach the point P, the secant line will begin to approach the tangent line.

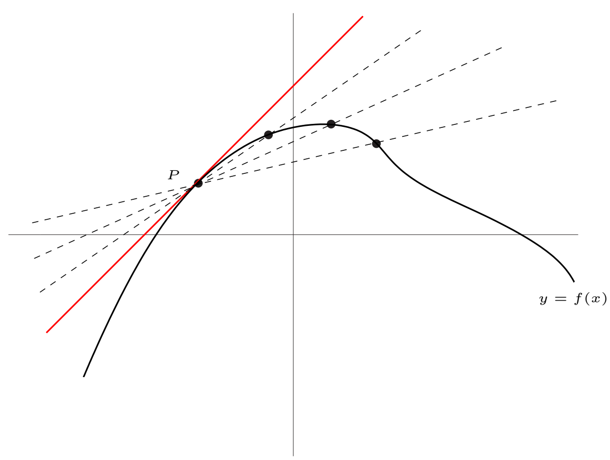


FIGURE 5. The secants approach the tangent as Q approaches P

So if we can study how the slope of the secant line changes as Q approaches P, we will have the value of the tangent line, as desired. This is what we'll talk about on Friday.