AN INTRODUCTION TO LIMITS

1. Announcements and Recap

A few standard announcements:

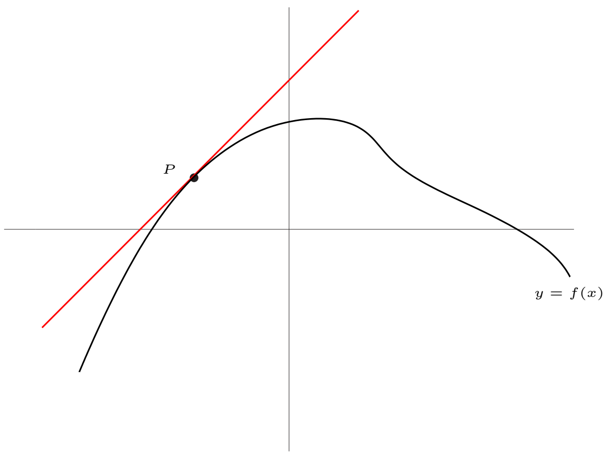
- Homework 2 will be posted by the end of the day
- Solutions to Quiz 1 and Homework 1 will be posted at the end of the day

As ever, today we'll be building on what we did in class last day. The highlights from last class include

- properties of logarithms (e.g., $\ln(1) = 0$);
- how to solve for the inverse of a composition of functions;
- how to graph $f^{-1}(x)$ given the graph of $f(x)$;
- writing equations of lines; and most important,
- an introduction to the tangent problem.

2. The Tangent Problem

Let's begin by reviewing the tangent problem. Recall that the tangent problem asks us to find the equation of the line tangent to a given function $f(x)$ at a given point on the graph $P = (x_0, f(x_0))$:



We said in class that in order to solve the tangent problem we needed to know a point on the tangent line and the slope of the tangent line. Since we're already provided with a point on the line (namely, the point the tangent line intersects the curve, $(x_0, f(x_0))$), all we need to know is the slope of the tangent line. How can we get this?

On Wednesday we said that we might try the following. After placing a random point $Q = (x_1, f(x_1))$ onto the graph of $f(x)$, we can write down the equation of the secant line through P and Q.

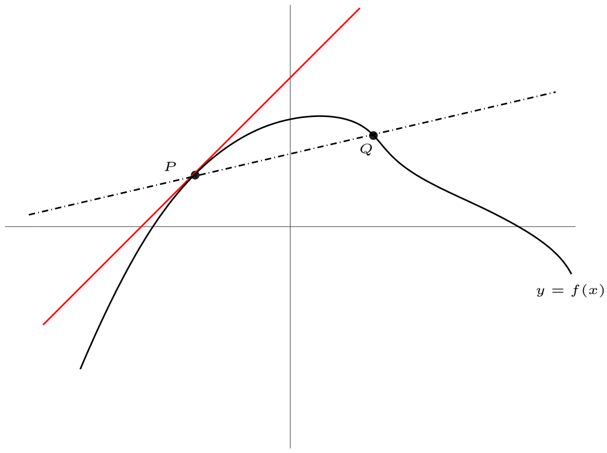


FIGURE 1. The graph of $f(x)$, the tangent, and the secant through P and Q

It will have slope

$$m = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Now comes the fun stuff: if we allow the point Q to slide toward the point P, the secant line through P and Q will begin to approach the tangent line.

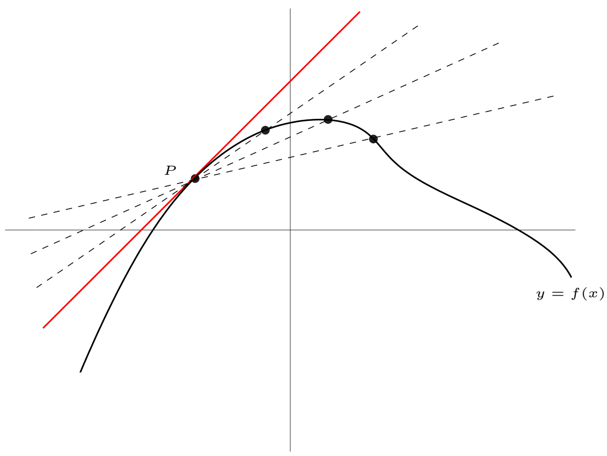


FIGURE 2. Secant lines approaching the tangent line

In particular, the slope of the secant line will start to approach the slope of the tangent line. Hence, if we can see what happens to the quantity

$$m = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

as Q approaches P, or equivalently as x_1 approaches x_0, we will have the slope of the tangent line as desired.

Soon we will be writing "what happens to the quantity $\frac{f(x_1)-f(x_0)}{x_1-x_0}$ as x_1 approaches x_0" as

$$\lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

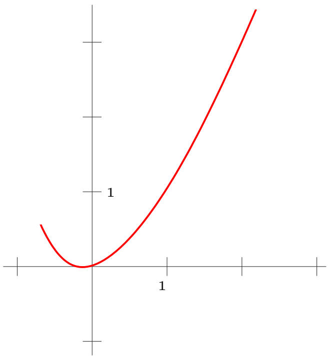
We will call this quantity *the derivative of* $f(x)$ *at* x_0, which we will abbreviate as $f'(x_0)$. In order to do this, however, we must first discuss the notion of a limit.

3. Limits

Intuitive Definition. For a function $f(x)$, the *limit of* f *as* x *approaches* a, written $\lim_{x\to a} f(x)$, is the quantity that outputs are approaching as inputs approach a.

In other words, $\lim_{x\to a} f(x)$ is where it looks like the graph of $f(x)$ is heading as x approaches a.

Example. Consider the graph of $f(x)$ below. What is $\lim_{x\to 2} f(x)$?

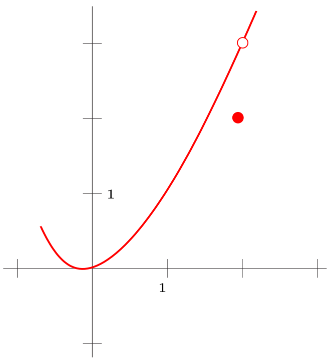


Solution. As inputs approach 2, outputs are approaching 3. Hence,

$$\lim_{x\to 2} f(x) = 3.$$

□

Example. Consider the graph of $g(x)$ below. What is $\lim_{x\to 2} f(x)$?

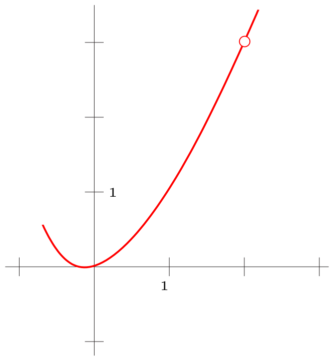


Solution. The function $g(x)$ is the same as $f(x)$ except at $x = 2$, where $g(x) = 2$. Notice, however, that in evaluating the limit we only care about where the function seems to be going as $x \to 2$, and not about the value of the function at 2. Hence we still have

$$\lim_{x\to 2} f(x) = 3.$$

□

Example. Consider the graph of $h(x)$ below. What is $\lim_{x\to 2} f(x)$?



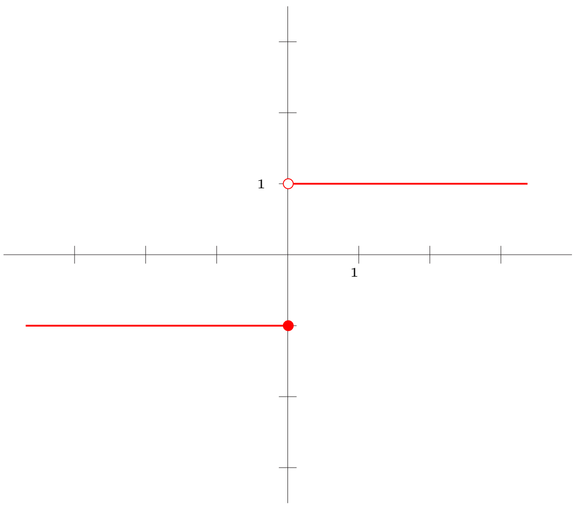
Solution. The function $h(x)$ is the same as $f(x)$ and $g(x)$ except at $x = 2$, where $h(x)$ isn't even defined. But again, computing the limit only requires that we look at where the function is headed as $x \to 2$ and not that we know anything about $h(2)$ (which in this case doesn't exist!), and so we have

$$\lim_{x\to 2} f(x) = 3.$$

□

The important lesson to take away from these examples is that the value of the limit $\lim_{x\to a} f(x)$ has (in general) nothing to do with the value $f(a)$.

Example. Consider the function $f(x)$ depicted below. What is $\lim_{x\to 0} f(x)$?



Solution. For this function, as inputs approach 0 from the left, outputs approach -1. But as inputs approach 0 from the right, outputs approach 1. Since there is not a single number that outputs approach as inputs approach 0, we say

$$\lim_{x\to 0} f(x) \text{ does not exist.}$$

$\square$

Because of this, it is convenient to talk about 'directional limits.'

Intuitive Definition. For a function $f(x)$, the *limit of f as x approaches a from the left*, written $\lim_{x\to a^-} f(x)$, is the quantity that outputs are approaching as inputs approach a from the left. The *limit of f as x approaches a from the right*, written $\lim_{x\to a^+} f(x)$, is the quantity that outputs are approaching as inputs approach a from the right.

Example. Consider the function $f(x)$ from the previous example. What is $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$?

Solution. We said above that $f(x)$ approaches 1 and -1 as inputs approach 0 from the right and left (respectively). Hence we have

$$\lim_{x\to 0^-} f(x) = -1 \text{ and } \lim_{x\to 0^+} f(x) = 1.$$

$\square$