CONTINUITY

1. Announcement

Midterm 2 is next Friday. I'll post a practice exam sometime Friday, and we'll have a review session Thursday night at 6:45. Also, a I've omitted a few problems from Homework 2 (namely problems 34, 35, and 47 from Section 2.4).

2. Recap

Last class period we saw some properties of limits which make it easy to compute the limit of a sum, difference, product, or quotient of two functions (under certain conditions). In particular, we saw

- how to use directional limits to determine if a function has a limit;
- when $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then the limit operator respects addition, subtraction, scaling, multiplication, and division (provided the denominator doesn't approach 0);
- an example of two functions which don't have a limit at 0, but whose sum does have a limit at 0;
- a preview of continuity.

Today we'll talk about continuity and how we can use it to evaluate limits. This will finally allow us to evaluate limits of complicated algebraic expressions, instead of just evaluating limits based on the graph of a function.

3. Continuity

In class on Monday we saw that evaluating the $\lim_{x\to a} f(x)$ is easy when $f(x)$ is a polynomial:

$$\lim_{x\to a} f(x) = f(a).$$

The class of functions which satisfy this condition are called *continuous functions.*

Definition. A function $f(x)$ is called continuous at a if

$$\lim_{x\to a} f(x) = f(a).$$

There are a few important things to notice about continuous functions:

(1) evaluating limits of continuous functions is easy, because it's the same as evaluating the function;
(2) this notion is probably different from the notion of continuity you saw in high school;
(3) a function can only be continuous on its domain, since the definition involves evaluating f at a (in particular if $f(a)$ isn't defined, then $f(x)$ can't be continuous at a);
(4) a continuous function has a limit at a (in particular, if $\lim_{x\to a} f(x)$ doesn't exist, f can't be continuous).

The really great news is that continuous functions are all around us! In fact, almost all the functions that we've talked about in our 'library' are continuous: polynomials, rational functions (on their domain–so in particular where the denominator isn't 0), algebraic functions (on their domain), trig functions (on their domain), exponential functions, log functions (on their domain), and the absolute value function are all continuous. We can also put together two continuous functions and get continuous functions: the sum, difference, product, scaling, quotient, and composition of two continuous functions are all continuous on the domain of the resultant function! This is great!

Example. The function $e^{\sqrt{\ln(x)}} - \cos(\sin(\pi x))$ is a continuous function on its domain, since it is written as the difference of functions which are themselves the composition of continuous functions. In particular this means

$$\lim_{x\to 1}\left[e^{\sqrt{\ln(x)}} - \cos(\sin(\pi x))\right] = e^{\sqrt{\ln(1)}} - \cos(\sin(\pi)) = e^0 - \cos(0) = 1 - 1 = 0.$$

Example. What is

$$\lim_{x\to 1}\left[\frac{x^{11}+\frac{1}{x}}{\sqrt{1-x+x^2}}\right]?$$

Solution. Since 1 is in the domain of our function and our function is the quotient of algebraic functions, the function is continuous at 1. This means we can evaluate the limit by plugging 1 into the equation. By the definition of continuity, we have

$$\lim_{x\to 1}\left[\frac{x^{11}+\frac{1}{x}}{\sqrt{1-x+x^2}}\right] = \frac{1+1}{0+1} = 2.$$

□

Tricky Example. Evaluate $\lim_{x\to 1} f(x)$ for the function

$$f(x) = \frac{x^2-1}{x-1}.$$

Solution. We would like to evaluate this limit by plugging in 1, but we see that 1 isn't in the domain of this function. Since this function isn't continuous at 1, we can't use our continuity trick (ie, plugging into the function) to evaluate this limit.

Nevertheless, we can use our incredible powers of cleverness to come up with a sneaky way to evaluate this limit. Indeed for all values $x \neq 1$ we have an equality

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1.$$

Since the function $g(x) = x+1$ is identical to $f(x)$ except at $x = 1$, we then have

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} g(x)$$

(the value of the limit doesn't depend on the value of the function at the limiting point!). But now we know that $\lim_{x\to 1} x+1 = 2$ since $x+1$ *is* continuous at 1, and so we have

$$\lim_{x\to 1} f(x) = 2.$$

□

This last trick is something we're going to use over and over and over again. We'll be interested in evaluating limits of functions at troublesome points, and we'll get around this by messing around with the function algebraically to find a new function which agrees with the old function everywhere except at the limiting point. The new function will be continuous, and so evaluating the limit will be a breeze.

4. Evaluating Limits of quotients

Remember the reason we began talking about limits is that we were interested in evaluating limits of the form

$$\lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Indeed, we're going to be interested in evaluating limits of quotients of functions from now on, so it's useful to have some guidelines. For the rest of this section we'll talk about evaluating a limit of the following form:

$$\lim_{x \to a} \frac{f(x)}{g(x)}.$$

We will always assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, since we will almost always be in that scenario.

Case 1: $\lim_{x\to a} g(x) \neq 0$. In this case, our limit rules tell us that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)},$$

and life is good. This will be your favorite thing to see, but sadly it won't happen often. We saw an example of this when we evaluated $\lim_{x\to 1} \dfrac{x^{11} + \frac{1}{x}}{\sqrt{1 - x + x^2}}$.

Case 2: $\lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} f(x) \neq 0$. In this case you can say with certainty that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \pm\infty.$$

Essentially the idea is that as $x \to a$, values in the numerator are approaching some fixed (nonzero) number, while numbers in the denominator are approaching 0. But if you take a fixed (nonzero) number and divide by a really tiny number, it's like taking that number and multiplying by an enormous number. As this enormous number gets larger and larger in magnitude, the product also gets larger and larger until it explodes.

Case 3: $\lim_{x\to a} g(x) = \lim_{x\to a} f(x) = 0$. This will be your absolute favorite case, because it means you get to do some detective work. As things stand, there's nothing you can say definitive about the limit, because (in an intuitive kind of way) the limit is looking like $\frac{0}{0}$, by far the most repugnant and evil force in the universe. In this case you always have to play some algebra tricks in order to simplify your quotient function, after which you can (hopefully) solve for the limit.

Example. Find

$$\lim_{x \to 1} \frac{\sqrt{9 - r} - 3}{r}.$$

Solution. We're going to use a trick that you probably remember from high school: we're going to multiply by 1 in a tricky way using the conjugate of the numerator. In particular, we have

$$\begin{aligned}
\lim_{r \to 0} \frac{\sqrt{9 - r} - 3}{r} &= \lim_{r \to 0} \frac{\sqrt{9 - r} - 3}{r} \cdot \frac{\sqrt{9 - r} + 3}{\sqrt{9 - r} + 3} \\
&= \lim_{r \to 0} \frac{(9 - r) - 9}{r(\sqrt{9 - r} + 3)} = \lim_{r \to 0} \frac{-r}{r(\sqrt{9 - r} + 3)} \\
&= \lim_{r \to 0} \frac{-1}{\sqrt{9 - r} + 3} = \frac{\lim_{r\to 0} -1}{\lim_{r\to 0} \sqrt{9 - r} + 3} = -\frac{1}{6}.
\end{aligned}$$

□

Example. Find

$$\lim_{t\to 0} \frac{\frac{t}{t+1} - t}{t}.$$

Solution. We're going to simplify the numerator by writing it as a common fraction. Then we'll play the normal cancellation game to evaluate the limit.

$$\begin{aligned}
\lim_{t\to 0} \left[\frac{\frac{t}{t+1} - t}{t}\right] &= \lim_{t\to 0} \left[\frac{\frac{t - t(t+1)}{t+1}}{t}\right] \\
&= \lim_{t\to 0} \left[\frac{t - t(t+1)}{t(t+1)}\right] \\
&= \lim_{t\to 0} \frac{t - t^2 - t}{t(t+1)} \\
&= \lim_{t\to 0} \frac{-t^2}{t(t+1)} = \lim_{t\to 0} \frac{-t}{t+1} = 0.
\end{aligned}$$

□

The examples we've seen are the three types you're most likely to see, and the technique for solving them is exactly the same as what we've shown here:

(1) if you have a quotient of polynomials, you need to factor the polynomials and get some cancellation;
(2) if you have a wonky square root, you want to multiply by the conjugate and get some cancellation; and
(3) if you have a difference of fractions in the numerator, you want to write that difference as a a single fraction, the proceed (probably by factoring).