THE INTERMEDIATE VALUE THEOREM

1. Announcements

Just a few announcements before we start.

(1) There is a box outside my office door that has old homeworks and quizzes. If you miss class on a day that I return these materials, or if for some reason I don't give yours back to you, you'll find your missing work in this box.
(2) We have a test next Friday, so your third homework assignment will be due *Wednesday*.

2. Recap

Last class period we talked about continuous functions. We began by giving a definition of continuity, and we said that continuous functions were great because they made evaluating limits really easy. We then had a long discussion about evaluating limits of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)},$$

and we established some criteria on how to evaluate limits that look like this. **This is very important**, so if you don't remember what we did you should look back at the old coursenotes.

3. More on continuity

We're going to start today by talking about a few miscellaneous topics on continuity, then we'll discuss the Intermediate Value Theorem.

3.1. Directional Continuity and Continuity on an interval.

First, just like when we were studying limits we discussed the notion of a directional limit, there is a version of continuity that is also directional.

Definition. A function $f(x)$ is said to be continuous from the left at a if

$$\lim_{x \to a^-} f(x) = f(a).$$

One could similarly define the notion of continuity from the right. (We'll see in a minute that the notion of directional continuity becomes important when discussing continuity on a closed interval.)

So far we've only talked about what it means for a function to be continuous at a point, but one can discuss continuity on a domain.

Definition. A function $f(x)$ is said to be continuous on the open interval (a, b) if it is continuous at every point c in the interval (a, b). A function $f(x)$ is said to be continuous on the closed interval $[a, b]$ if it is continuous on (a, b), continuous from the right at a, and continuous from the left at b.

3.2. **Types of discontinuity.** A function can fail to be continuous in a few different ways. The two big ways we'll see for a function to fail to be continuous at a point are jump discontinuities and removable discontinuities. A jump discontinuity is a point a so that $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist, but are not equal. A removable discontinuity is a point a so that $\lim_{x\to a} f(x)$ exists, but $\lim_{x\to a} f(x) \neq f(a)$.

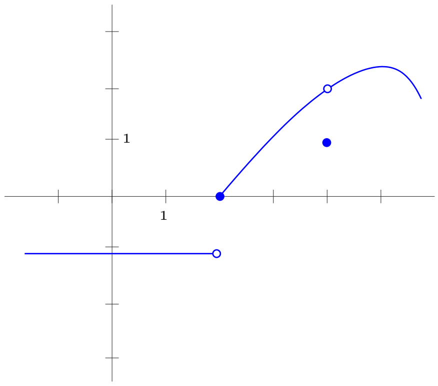

FIGURE 1. A removable and a jump discontinuity

4. Intermediate Value Theorem

Continuous functions have another nice property which is described by the intermediate value theorem.

Intermediate Value Theorem. If $f(x)$ is a continuous function on $[a, b]$ and either

- $f(a) < N$ and $f(b) > N$ or
- $f(a) > N$ and $f(a) < N$,

then there exists a number $c \in (a, b)$ so that $f(c) = N$.

This theorem is incredibly useful for proving results like the following:

Example. Show that the equation $x^{2006} + 2006x - 1$ has a root.

Solution. A root of the equation $f(x) = x^{2006} + 2006x - 1$ is a solution to the equation $f(x) = 0$, so we have to prove that $f(x) = 0$ has a solution.

First, we notice that $f(x)$ is a continuous function, since $f(x)$ is a polynomial, and polynomials are continuous. Further we can evaluate

$$f(0) = 0^{2006} + 2006 \cdot 0 - 1 = -1 < 0$$
$$f(1) = 1^{2006} + 2006 \cdot 1 - 1 = 1 + 2006 - 1 = 2006 > 0.$$

Since $f(x)$ is a continuous function on $[0, 1]$ with $f(0) < 0$ and $f(1) > 0$, we may apply the intermediate value theorem and conclude that there exists a number c in the interval $(0, 1)$ with $f(c) = 0$. This is exactly what we were after! □

NB: Our same argument shows, for instance, that $x^{2006} + 2006x - 1 = 1000$ has a solution in $(0, 1)$: $f(x)$ is still continuous on $[0, 1]$, and by our calculation we have $f(0) = -1 < 1000$ and $f(1) = 2006 > 1000$; the intermediate value theorem says there exists some $\tilde{c}$ with $\tilde{c}^{2006} + 2006\tilde{c} - 1 = 1000$.

When one is first introduced to the intermediate value theorem, it seems hard to know when you should be using it to solve a problem. Here's a great thing to keep in mind:

Rule of Thumb. If someone asks you to show there exists a number with some nice property but doesn't ask you to explictly produce that number (i.e., to tell them an exact value), you're almost certainly going to use the intermediate value theorem.

Indeed, the conclusion of the intermediate value theorem is somewhat mysterious in that it states the existence of a number c which has a nice property $f(c) = N$ but doesn't actually tell you what that number is. This is about the only rule we have to show that a certain number exists without actually producing it, so if someone asks you to show a certain number exists but doesn't make you actually find it, you should start thinking 'intermediate value theorem.'

Here's a trickier example that is a good example of this principle.

Example. Show that there exists a number x in the interval $[0, \frac{\pi}{2}]$ with $x = \cos(x)$.

Solution. This problem asks us to show there exists a number x which has a 'nice property' (this time, 'nice property' means $x = \cos(x)$) but doesn't ask us to say exactly what it is, so we're thinking that we should use the intermediate value theorem.

The first thing we need to do is translate this problem into a problem that we can use the intermediate value theorem to solve. In particular, given a continuous function $g(x)$ and some hypotheses, the intermediate value theorem let's us conclude the existence of a number c with $g(c) = N$, where N is some real number. So let's translate our problem into the solution of an equation.

Specifically, we notice that finding a number x with $x = \cos(x)$ is the same as finding a solution to the equation $f(x) = 0$, where $f(x) = \cos(x) - x$. Why is this true? A solution to $f(x) = 0$ is a number so that $\cos(x) - x = 0$, or equivalently, so that $\cos(x) = x$. This is exactly the condition we want!

Great, so let's try to show that for the function $f(x) = \cos(x) - x$ there exists a solution to $f(x) = 0$. Since we know we have to use the intermediate value theorem, we start by observing that $f(x)$ is the difference of two continuous functions, and is therefore itself continuous. Furthermore we can see that

$$f(0) = \cos(0) - 0 = 1 - 0 = 1 > 0$$
$$f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) - \frac{\pi}{2} = 0 - \frac{\pi}{2} = -\frac{\pi}{2} < 0.$$

Therefore, since $f(x)$ is a continuous function on $[0, \frac{\pi}{2}]$ with $f(0) > 0$ and $f(\frac{\pi}{2}) < 0$, we may apply the intermediate value theorem and conclude that there exists a number c in the interval $(0, \frac{\pi}{2})$ with $f(c) = 0$. This magical value of c therefore satisfies $\cos(c) - c = 0$, and so $\cos(c) = c$. That's just what we want. $\square$