# THE DERIVATIVE AT A POINT

## 1. Announcements

Since we have a test on Friday and homework for this week has been pushed forward to Wednesday, I'll be holding extra office hours. I'll be around Monday from 11-12 and again from 12:45 to 2:15, then Tuesday I'll be in my office from 11-12 and again from 1-2. You should feel free to drop by and ask questions about whatever you like. As always, you can also stop by at times other than those, though if you don't email me to let me know you run the risk of my absense.

## 2. Final thoughts on the Intermediate Value Theorem

In class on Friday we talked about the intermediate value theorem, and today I wanted to do one more example using this tool. It is one of the Practice Problems from last class period.

**Example.** Show that there exists a number which is 1 less than its cube.

*Solution.* In class on Friday we said that when one reads a problem that starts 'Show there exists a number so that...' in which the problem doesn't actually ask you to produce said number, it's a great sign that one should use the intermediate value theorem. That's the situation we find ourselves in for this problem, so we know we'll somehow be using the intermediate value theorem.

With that in mind, we begin by translating this statement into an equation that we can use. Indeed, to say that a number is 1 less than its cube is to say that the number is a solution to the equation

$$x = x^3 - 1.$$

Since the intermediate value theorem lets us conclude when a given function achieves a certain numerical value, we re-translate this equation into the form $f(x) = 0$ for some $f(x)$. In this case, a solution to the equation $x = x^3 - 1$ is the same as a solution to the equation

$$x^3 - x - 1 = 0$$

(we've just moved the $x$ onto the other side of the equation). This is great, because we can use the intermediate value theorem to find a solution to this equation: all we have to do is verify the function on the left-hand side is continuous and that it takes on a value less than 0 and a value greater than 0.

We begin by noting that $x^3 - x - 1$ is a polynomial, and hence is continuous everywhere. Now we're on the hunt for numbers $a$ and $b$ with $f(a) < 0$ and $f(b) > 0$. We'll try some values that are particularly easy to evaluate:

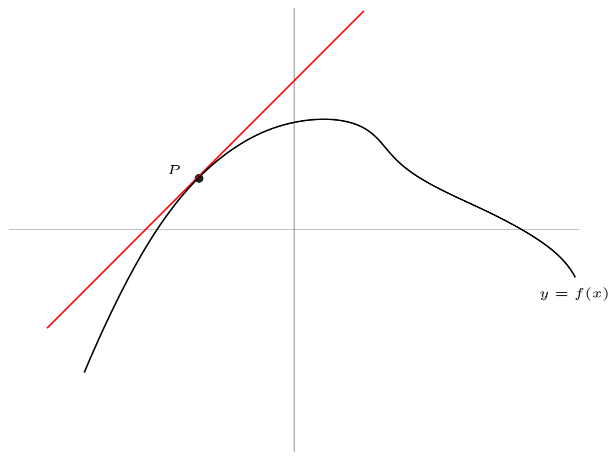
$$\begin{aligned} f(0) &= 0 - 0 - 1 = -1 < 0 \\ f(1) &= 1 - 1 - 1 = -1 < 0 \\ f(2) &= 8 - 2 - 1 = 5 > 0. \end{aligned}$$

Great! Now we can use the intermediate value theorem. Since $x^3 - x - 1$ is continuous on $[0, 2]$, and $f(0) < 0$ and $f(2) > 0$, the intermediate value theorem tells us there exists a number $c$ in $(0, 2)$ which is a solution to $x^3 - x - 1 = 0$. This is exactly what we want. □

## 3. Derivative at a point

Having explored limits and continuous functions to their full, we can now go back to the tangent problem and solve it. This is exciting! The tangent problem has been around for thousands of years and wasn't resolved until a few hundreds years ago. It took the mind of Newton and Liebenisz to formulate a precise solution, and we're about to do it ourselves. Be proud.

Let's begin by reviewing the tangent problem. Recall that the tangent problem asks us to find the equation of the line tangent to a given function $f(x)$ at a given point on the graph $P = (x_0, f(x_0))$:




We said in class that in order to solve the tangent problem we needed to know a point on the tangent line and the slope of the tangent line. Since we're already provided with a point on the line (namely, the point the tangent line intersects the curve, $(x_0, f(x_0))$), all we need to know is the slope of the tangent line. How can we get this?

A few class periods ago we said that we might try the following. After placing a random point $Q = (x_1, f(x_1))$ onto the graph of $f(x)$, we can write down the equation of the secant line through $P$ and $Q$.

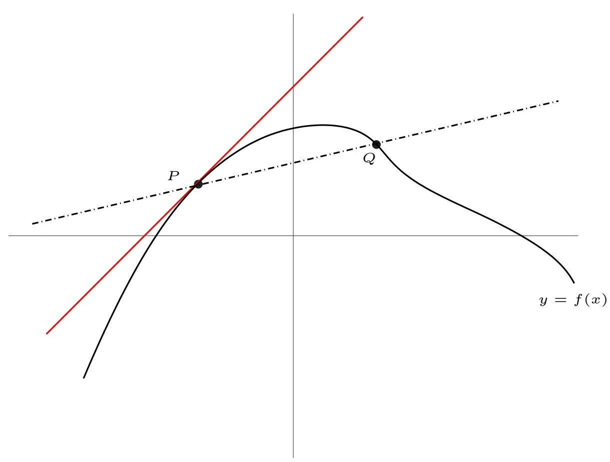



Figure 1. The graph of $f(x)$, the tangent, and the secant through $P$ and $Q$

It will have slope

$$m = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Now comes the fun stuff: if we allow the point $Q$ to slide toward the point $P$, the secant line through $P$ and $Q$ will begin to approach the tangent line.

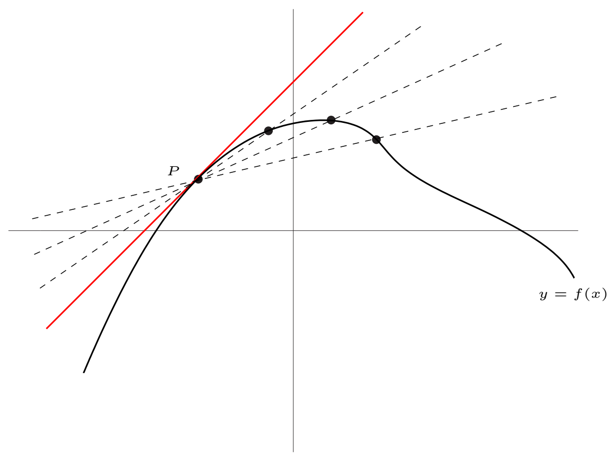



FIGURE 2. Secant lines approaching the tangent line

In particular, the slope of the secant line will start to approach the slope of the tangent line. Hence, if we can see what happens to the quantity

$$m = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

as $Q$ approaches $P$, or equivalently as $x_1$ approaches $x_0$, we will have the slope of the tangent line as desired. Since we now know how to evaluate limits, we can determine the slope of the tangent line, and therefore solve the tangent problem.

**Example.** Solve the tangent problem for the function $f(x) = x^2$ at the point $(1, 1)$.

*Solution.* The tangent problem asks us to find the equation of the line tangent to $f(x)$ at the point $(1, 1)$. Since writing the equation of a line requires knowing a point on the line and the slope of the line, and since we already know a point on the line (namely, $(1, 1)$), we know the line will look like

$$y - 1 = m(x - 1),$$

where here $m$ is the slope of the tangent line at $(1, 1)$. (Note: We used slope-point form to write the equation of the line in this way.)

Great, so all we need to know now is $m$. We've said that $m = \lim_{x_1 \to x_0} \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$, where in this case $x_0 = 1$. Renaming $x_1$ as $x$, we evaluate:

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x+1)(x-1)}{x - 1} = \lim_{x \to 1} x + 1 = 1 + 1 = 2.$$

Since we know $m = 2$, we plug this into our previous expression for the line to get the final equation:

$$y - 1 = 2(x - 1).$$

□

**Example.** Solve the tangent problem for the function $f(x) = x^2$ at the point $(2, 4)$.

*Solution.* We're solving the tangent problem for the same function, but now at a new point. Our overall strategy will be the same.

Since we know $(2, 4)$ is a point on the tangent line, the equation of the tangent line will look like

$$y - 4 = m(x - 2),$$

where here $m$ is the slope of the tangent line at $(2, 4)$. (Again, we used slope-point form to write the equation of the line in this way.)

So we need to know now is $m$, which (like last time) is $\lim_{x\to 2} \dfrac{f(x) - f(2)}{x - 2}$. We evaluate:

$$m = \lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2} \frac{x^2 - 4}{x - 2} = \lim_{x\to 2} \frac{(x + 2)(x - 2)}{x - 2} = \lim_{x\to 2} x + 2 = 2 + 2 = 4.$$

Since $m = 4$, we can plug this into our previous expression for the line to get the final equation:

$$y - 4 = 4(x - 2).$$

□

When you move out into the wider mathematical world, you will find that people evaluate the slope of the tangent line in a few different, but equivalent, ways. We've seen already that

$$\text{slope of tangent to } f(x) \text{ at } (x_0, f(x_0)) = \lim_{x_1\to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Sometimes instead of using the name $x_0$ people will use $a$, and they will say $x$ instead of $x_1$. Then we have

$$\text{slope of tangent to } f(x) \text{ at } (a, f(a)) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

Then again some people will let $h = x - a$ represent the distance between $x$ and $a$. Since this gives $x = a + h$, you can substitute in the previous expression and get

$$\text{slope of tangent to } f(x) \text{ at } (a, f(a)) = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h}.$$

*All three of these expressions are equivalent ways to compute the slope of the tangent line, and you should feel comfortable using at least one of them.* My preference is for the latter, though we'll definitely use the second one as well. The first is good for motivating ideas, but not so good in computations (since there are so many subscripts), so we won't use it in practice much.

**Example.** We've said that we can compute the slope of the tangent line in two ways. For the function $f(x) = x^2$ and the point $(2, 4)$, show these two methods agree.

*Solution.* In the previous example we saw that $\lim_{x\to 2} f(x) - f(2)x - 2 = 4$. Now we compute the other limit:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(2 + h) - f(2)}{h} &= \lim_{h\to 0} \frac{(2 + h)^2 - 2^2}{h} = \lim_{h\to 0} \frac{4 + 4h + h^2 - 4}{h} \\
&= \lim_{h\to 0} \frac{4h + h^2}{h} = \lim_{h\to 0} \frac{h(4 + h)}{h} = \lim_{h\to 0} 4 + h = 4.
\end{aligned}$$

Happily, both limits give the same value. □

**Definition.** For a function $f(x)$, the derivative of $f(x)$ at $a$, written $f'(a)$, is defined to be

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a},$$

if it exists. If this limit does not exist, we say that $f(x)$ fails to be differentiable at $a$ or that $f'(a)$ is undefined. The derivative $f'(a)$ represents the slope of the line tangent to $f$ at $a$. From the comments above, we see that we can also evaluate $f'(a)$ by computing a seemingly different limit:

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

**Example.** Find the equation of the line tangent to $f(x)=\frac{1}{x}$ at $a=3$.

*Solution.* We're asked to find the equation of the line tangent to $f(x)$ at the point $(3, f(3)) = (3, \frac{1}{3})$. Just like before, we know this equation will take the form

$$y-\frac{1}{3}=m(x-3),$$

where here $m$ is the slope of the line tangent to $f(x)$ at $(3, \frac{1}{3})$.

How can we compute $m$? We compute $f'(3)$:

$$\begin{aligned} f'(3) &= \lim_{x\to 3}\frac{f(x)-f(3)}{x-3} = \lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{3}}{x-3} = \lim_{x\to 3}\frac{\left(\frac{3-x}{3x}\right)}{x-3} \\ &= \lim_{x\to 3}\frac{3-x}{(x-3)(3x)} = \lim_{x\to 3}\frac{-(x-3)}{(x-3)(3x)} = \lim_{x\to 3}\frac{-1}{3x} = -\frac{1}{9}. \end{aligned}$$

With this in mind, we can finish writing the equation of the tangent line:

$$y-\frac{1}{3}=-\frac{1}{9}(x-3).$$

□