THE DERIVATIVE AS A FUNCTION

1. Recap

So far in the course our motivation has been the tangent problem: for a given function $f(x)$ and point $(a, f(a))$, find the equation of the line tangent to $y = f(x)$ at $(a, f(a))$. This problem led us to investigate limits, which we then used to calculate the slope of the line tangent to $y = f(x)$ at $(a, f(a))$. We called this value the derivative of f at a, written $f'(a)$, and said it was

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Our focus will shift slightly now. Instead of just solving the tangent problem at a fixed point $(a, f(a))$, we will be interested in calculating the derivative of a function for all points at once. Later, we'll use this information to solve more interesting problems.

2. Derivatives

So far we have considered the derivative of a function *at a fixed point* a. Today, we change our perspectice slightly and instead speak of the derivative as a function itself. For a given input x, this derivative function will return for us the derivative of f at the point x. Explicitly

Definition. For a function $f(x)$, the derivative of $f(x)$, written $f'(x)$ or $\frac{d}{dx}[f(x)]$, is defined to be

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

if it exists. The derivative $f'(x)$ is the function which gives the slope of the line tangent to f at a point x.

Notice that in particular this derivative function is very much like the limits we have been considering in the past few weeks. The only difference is that instead of having an explicit value for x in the limit above, we leave this quantity as a variable. This means that, generally speaking, the derivative $f'(x)$ will be a function of x and not just a number.

Let's do an example.

Example. Use the definition of the derivatice to compute $f'(x)$ where $f(x) = x^2$.

Solution. Using the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0} 2x + h = 2x. \end{aligned}$$

□

Just like when we computed derivatives last week, we can use geometry to make computations a bit easier.

Example. Using only that that derivative $f'(a)$ is the slope of the tangent line to f at a, find the derivative of $f(x) = c$.

Solution. The graph of the function $f(x) = c$ is a horizontal line which passes through the point $(0, c)$. What is the tangent to this graph at a point on the graph? One can see this tangent line is another horizontal line, which of course has slope 0. This means that for any point x we want, the slope of the tangent line through f at x is 0. In other words, for all x we have $f'(x) = 0$.

We could also compute this derivative using the definition. Let's try it out.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0.$$

□

Example. Using only that that derivative $f'(a)$ is the slope of the tangent line to f at a, find the derivative of $f(x) = mx + b$.

Solution. The graph of the function $f(x) = mx + b$ is a line which passes through the point $(0, b)$ and has slope m. What is the tangent to this graph at a point on the graph? One can see this tangent line is just the line $y = mx + b$ again, which of course has slope m. This means that for any point x we want, the slope of the tangent line through f at x is m. In other words, for all x we have $f'(x) = m$.

We could also compute this derivative using the definition. Let's try it out.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{m(x+h) + b - (mx+b)}{h} \\ &= \lim_{h\to 0} \frac{mx + mh + b - mx - b}{h} = \lim_{h\to 0} \frac{mh}{h} = \lim_{h\to 0} m = m. \end{aligned}$$

□

Example. Use the graph of $f(x) = \sin(x)$ to sketch $f'(x)$; does this function look familiar?

Solution. The graph of $\sin(x)$ looks like the following:

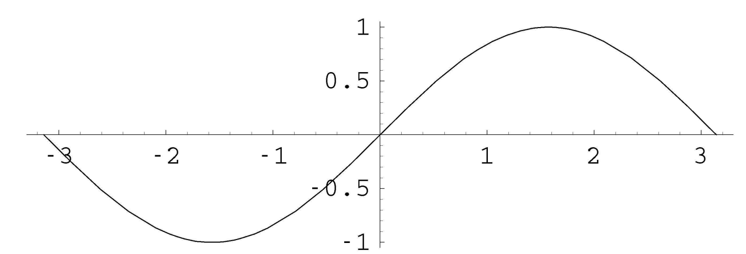

Figure 1. The sine function

Using the graph of $\sin(x)$, we see first that $f'(x)$ is 0 for $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$, since at these points the tangent lines are horizontal. Hence, we can plot these values on the graph of $f'(x)$ straightaway.

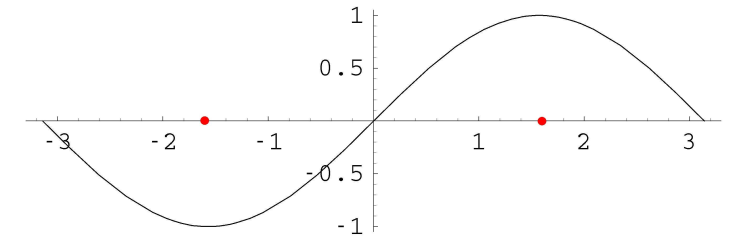

FIGURE 2. The sine function, with the value of $\frac{d}{dx}[\sin(x)]$ at $\pm\frac{\pi}{2}$ plotted in red

What might we try next? If we look at $x = 0$, it appears that the slope of the tangent line is 1. (To see this, try sketching the line $y = x$ onto this graph. It should appear tangent to the graph at 0.) Hence we can believe $f'(0) = 1$. Similarly, we can 'eyeball' that the slope of the tangent line through $x = -\pi$ is -1 and that the slope of the tangent line through $x = \pi$ is also -1. So, we can plot $f'(0) = 1$ and $f'(-\pi) = f'(\pi) = -1$.

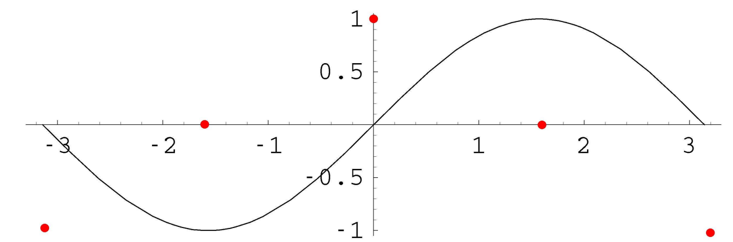

FIGURE 3. The sine function, with a few more points of $\frac{d}{dx}[\sin(x)]$ plotted in red

Using these 5 points, and noticing that the slope of the tangent stays fairly close to 1 near 0 and fairly close to -1 near $-\pi$ and π, it is not unthinkable that $f'(x)$ look something like this:

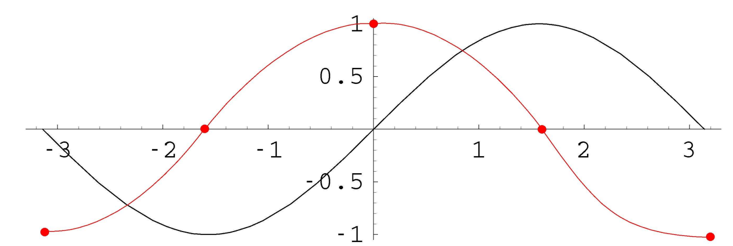

FIGURE 4. The sine function, with a 'rough' sketch of $\frac{d}{dx}[\sin(x)]$ in red

$\square$

Steve pointed out in class that we can see this is a reasonable graph of $f'(x)$ for the following reason. After plopping a random point on the graph of $f(x)$, check to see whether the slope of the tangent line at that point is positive or negative. If it's positive, you'll see that our graph of $f'(x)$ is also positive at that point; if it's negative, the graph of $f'(x)$ we've drawn will be negative at that point. This is a handy thing to keep in mind.

In fact, given the graph of $f'(x)$ we can attempt to sketch a graph of $f(x)$. I showed you how you can do this in class, and instead of reproducing the sketches here I think it's better for you to come talk to me if this doesn't make sense. We can go through some examples to make this clear. These examples are important because, just like when we computed derivatives last week, the interplay between geometry and algebra can be important in understanding the derivative of a function.

Let's do another example.

Example. Compute $\frac{d}{dx}\left[\sqrt{x}\right]$.

Solution. This question is asking the following: for the function $f(x) = \sqrt{x}$, what is $f'(x)$. We compute this using the definition, and then following the same steps we developed in the last couple of weeks (in this case, for instance, the key will be multiplying by the conjugate).

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \lim_{h\to 0} \left[\frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)\right]$$

$$= \lim_{h\to 0} \frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

□

Steve posed the following very good question.

Example. Compute $\frac{d}{dx}\left[x+\sqrt{x}\right]$.

Solution. As before, this question is asking the following: for the function $f(x) = x+\sqrt{x}$, what is $f'(x)$. To compute this limit we're going to be a little sneaky, since multiplying by the conjugate would get wildly messy.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{x+h+\sqrt{x+h}-(x+\sqrt{x})}{h} = \lim_{h\to 0} \frac{(x+h-x)+\left(\sqrt{x+h}-\sqrt{x}\right)}{h}$$

$$= \lim_{h\to 0} \left[\frac{(x+h-x)}{h} + \frac{\sqrt{x+h}-\sqrt{x}}{h}\right] = \lim_{h\to 0} \left[\frac{(x+h-x)}{h}\right] + \lim_{h\to 0} \left[\frac{\sqrt{x+h}-\sqrt{x}}{h}\right].$$

Now we recognize that the first summand is just $\dfrac{d}{dx}[x]$, which by a previous computation we know is just 1. The second summand is $\dfrac{d}{dx}\left[\sqrt{x}\right]$, and by the previous example this is $\dfrac{1}{2\sqrt{x}}$. Hence we have

$$\frac{d}{dx}\left[x+\sqrt{x}\right] = \lim_{h\to 0} \left[\frac{(x+h-x)}{h}\right] + \lim_{h\to 0} \left[\frac{\sqrt{x+h}-\sqrt{x}}{h}\right] = 1 + \frac{1}{2\sqrt{x}}.$$

□

We'll see this is a manifestation of a nice property of derivatives: the derivative of a sum is the sum of derivatives. We'll make this precise next time.