THE DERIVATIVE AS A FUNCTION, PART II

1. Announcements

There are only a few announcements this time

- I'm going to cancel the problems from section 2.9 from your homework, so only work the problems from section 2.8.
- Also, on the last homework I noticed that some of you used a calculator on problems where you weren't authorized to use a calculutor. Shame! Next time I'll deduct points.

2. Recap

Last time in class we talked about the derivative of a function as a function itself. For a function $f(x)$ we said the derivative can be written either as $f'(x)$ or $\frac{d}{dx}[f(x)]$, and is given by

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

We computed the derivative of several of our favorite functions. For example, we saw

- $\frac{d}{dx}[mx+b]=m,$
- $\frac{d}{dx}\left[x^2\right]=2x,$
- $\frac{d}{dx}[\sqrt{x}]=\frac{1}{2\sqrt{x}}.$

using the definition of the derivative. We also talked about the geometry of the derivative: using the graph of a function to sketch the graph of its derivative, and the graph of the derivative to sketch the graph of the original function.

Today I want to talk about how a function can fail to be differentiable, compute a few more (hard) examples using the definition of the derivative, and then introduce a shortcut that will make computing derivatives much easier.

3. Non-differentiability

We say that a function f is differentiable at a if the derivative of f is defined at a. That is to say, f is differentiable at a if the following limit exists:

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

Although we will be dealing mostly with differentiable functions in this class, there are a handful of times we will encounter functions which have points which are not differentiable. These are points a so that

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \quad \text{does not exist.}$$

Although there are many ways a function could fail to be differentiable at a point a, there are three typical types of non-differentiability.

(1) **Cusps.** A function is not differentiable at a point where the graph of f has a kink or corner (one might also call such a point on the graph a kink). Essentially, these places fail to be differentiable because the left and right hand limits

$$\lim_{h\to 0^-}\frac{f(a+h)-f(a)}{h} \quad \text{and} \quad \lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}$$

do not match up. For instance, the absolute value function $f(x)=|x|$ fails to be differentiable at 0 because

$$\lim_{h\to 0^-}\frac{f(a+h)-f(a)}{h}=-1$$

and

$$\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}=1.$$

(2) **Discontinuities.** A function is not differentiable at a point where the graph of f is not continuous. In this case, the limit

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

does not exist, because the denominator approaches 0 as $h\to 0$, while the numerator approaches some finite, nonzero number (remember that since f is not differentiable at a, we have

$$\lim_{h\to 0} f(x)\neq f(a),$$

and hence $\lim_{h\to 0} f(a+h)-f(a)\neq 0$).

(3) **Vertical Tangents.** Finally, a function is not differentiable at a point on the graph where the tangent line to f is a vertical line. This is because the slope of the tangent to the graph at this point is infinite, which in our class corresponds to 'does not exist.'

Example. The following function displays all 3 failures of differentiability. Notice that there are two places where the function fails to be continuous (one a jump discontinuity and one a removable discontinuity).

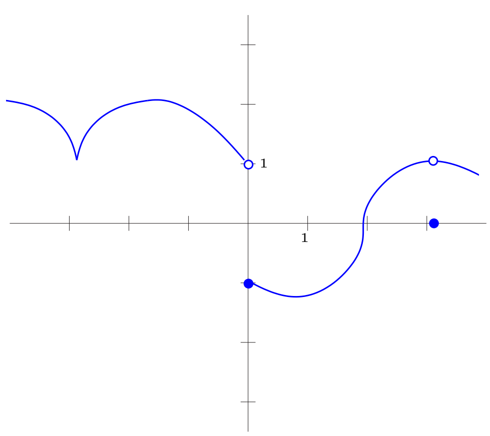

FIGURE 1. A function which fails to be differentiable at 4 points

4. A few more computations

We're going to compute a few more examples, but this time we're going to do some tricky functions. You won't be expected to recreate these arguments anywhere, but it's good to see this (admittedly painful) examples.

Example. Compute $\frac{d}{dx}[\sin(x)]$ using

$$\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h), \quad \lim_{h\to 0}\frac{\sin(h)}{h} = 1, \quad \text{and} \quad \lim_{h\to 0}\frac{\cos(h)-1}{h} = 0.$$

Solution. This example won't use any spectacular ideas except for the facts above. It is our job to put our limit in a form where we can use these facts. Anyway, let's proceed

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} = \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h} = \lim_{h\to 0}\left(\frac{\sin(x)(\cos(h)-1)}{h} + \frac{\cos(x)\sin(h)}{h}\right)$$

$$= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)}{h} + \lim_{h\to 0}\frac{\cos(x)\sin(h)}{h}$$

$$= \left[\lim_{h\to 0}\sin(x)\right]\left[\lim_{h\to 0}\frac{\cos(h)-1}{h}\right] + \left[\lim_{h\to 0}\cos(x)\right]\left[\lim_{h\to 0}\frac{\sin(h)}{h}\right] = \sin(x)\cdot 0 + \cos(x)\cdot 1 = \cos(x)$$

$\square$

Example. Compute $\frac{d}{dx}[\sqrt[3]{x}]$.

Solution. So far we've only done a handful of different types of limits. This one is a bit different, and requires a new 'trick.' You won't be expected to improvise this kind of trick, so don't be too scared by this example.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}\,\frac{\sqrt[3]{(x+h)^2}+\sqrt[3]{x(x+h)}+\sqrt[3]{x^2}}{\sqrt[3]{(x+h)^2}+\sqrt[3]{x(x+h)}+\sqrt[3]{x^2}}$$

$$= \lim_{h\to 0}\frac{\sqrt[3]{(x+h)^3}+\sqrt[3]{x(x+h)^2}+\sqrt[3]{x^2(x+h)}-\sqrt[3]{x(x+h)^2}-\sqrt[3]{x^2(x+h)}-\sqrt[3]{x^3}}{h(\sqrt[3]{(x+h)^2}+\sqrt[3]{x(x+h)}+\sqrt[3]{x^2})}$$

$$= \lim_{h\to 0}\frac{x+h-x}{h(\sqrt[3]{(x+h)^2}+\sqrt[3]{x(x+h)}+\sqrt[3]{x^2})} = \lim_{h\to 0}\frac{h}{h(\sqrt[3]{(x+h)^2}+\sqrt[3]{x(x+h)}+\sqrt[3]{x^2})}$$

$$= \lim h \to 0\frac{1}{\sqrt[3]{(x+h)^2}+\sqrt[3]{x(x+h)}+\sqrt[3]{x^2}} = \frac{1}{\sqrt[3]{x^2}+\sqrt[3]{x^2}+\sqrt[3]{x^2}} = \frac{1}{3\sqrt[3]{x^2}}.$$

$\square$

5. Derivative Shortcuts

The previous two examples where there to scare you into realizing that computing derivatives using the definition can get really nasty really fast. Even a function as simple as $\sqrt[3]{x}$ required a real flash of inspiration to compute. Happily, there are some rules that let us evaluate derivatives more easily. The first two we could prove using the definition of the derivative as a limit.

5.1. Linear Properties of the Derivative.

We know that given two functions $f(x)$ and $g(x)$ we can build new functions like $f(x)+g(x)$ or $f(x)g(x)$. How do the derivatives of these new functions relate to the derivatives of the old functions? In the case of addition, the rule is pretty simple:

$$\frac{d}{dx}\left[f(x)+g(x)\right] = \frac{d}{dx}\left[f(x)\right] + \frac{d}{dx}\left[g(x)\right].$$

With your friends you would describe this rule as 'the derivative of a sum is the sum of derivatives.' This trick is actually quite useful in evaluating derivatives. Happily, this is also what we would want the derivative of a sum to be (we won't be so lucky with products!).

The derivative also behaves nicely with respect to scalar multiplication. Namely, if c is a constant (like 4 or 11 or π), then

$$\frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}\left[f(x)\right].$$

5.2. The Power Rule.

This next rule is going to be your best friend in the rest of the class. You'll use it so much that you'll wonder how you ever lived without it!

Power Rule. If n is a number, then

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}.$$

Example. Using the power rule we have

$$\frac{d}{dx}\left[x^2\right] = 2\cdot x^{2-1} = 2x^1 = 2x$$

$$\frac{d}{dx}\left[\sqrt{x}\right] = \frac{d}{dx}\left[x^{\frac{1}{2}}\right] = \frac{1}{2}\cdot x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left[\sqrt[3]{x}\right] = \frac{d}{dx}\left[x^{\frac{1}{3}}\right] = \frac{1}{3}\cdot x^{\frac{1}{3}-1} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}} = \frac{1}{3\sqrt[3]{x^2}}.$$

This was way easier than computing a limit! It also agrees with computations we've already seen.

Example. Use the rules above to compute $\frac{d}{dx}\left[x^{11}+2x^2+x^{-1}\right]$.

Solution. We begin by splitting the derivative across the sum, then pull out the scalar 2 in the second summand, then use the power rule. We get

$$\frac{d}{dx}\left[x^{11}+2x^2+x^{-1}\right] = \frac{d}{dx}\left[x^{11}\right] + \frac{d}{dx}\left[2x^2\right] + \frac{d}{dx}\left[x^{-1}\right] = \frac{d}{dx}\left[x^{11}\right] + 2\frac{d}{dx}\left[x^2\right] + \frac{d}{dx}\left[x^{-1}\right]$$

$$= 11x^{11-1} + 2(2x^{2-1}) + (-1)x^{-1-1} = 11x^{10} + 4x - x^{-2}$$

□