DERIVATIVES OF EXPONENTIALS AND TRIGS

1. A question before class

Tavita asked a very good question just before class started. We said in class last day that there were three (typical) ways a function could fail to be differentiable at a point: cusps, discontinuities, and vertical tangent lines. Now Tavita asks: how can a function have a vertical tangent line and still be a function? After all, we have the vertical line test which tells us whether a graph is the graph of a function. It would seem that if a tangent line is vertical than the function would fail the vertical line test at this point, and hence not be a function.

It turns out that a function can have a vertical tangent line and still be a function. The prototypical example of such a function is $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$. The graph of $f(x)$ looks like this, and you can see that it has a vertical tangent at 0.

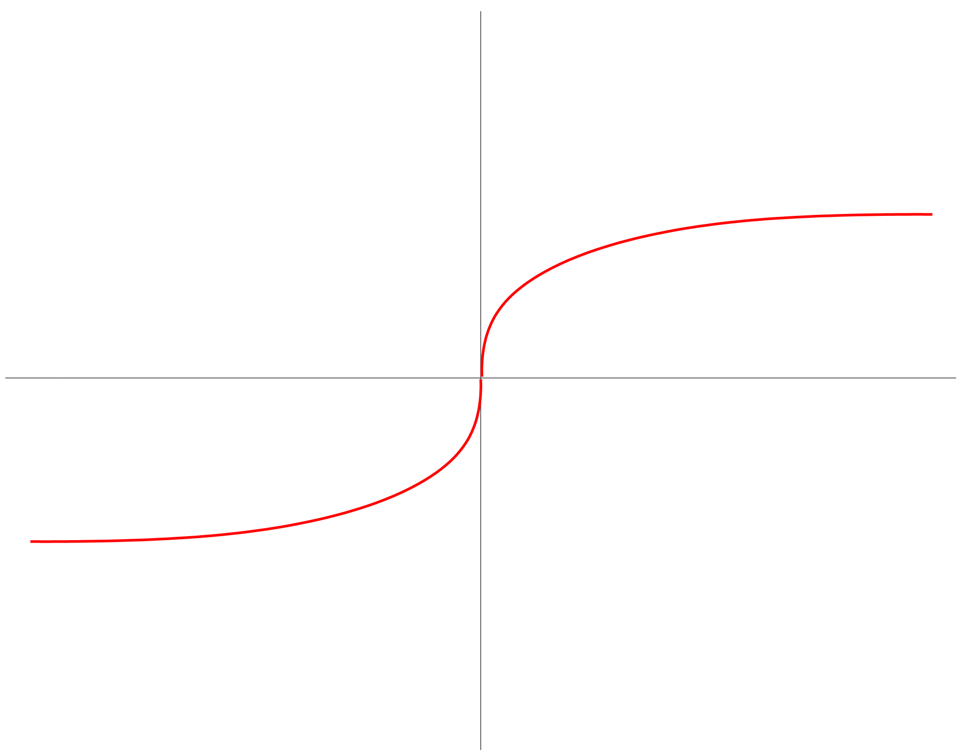

Figure 1. The graph of $f(x) = x^{\frac{1}{3}}$

But this is still a function, because there is only one value for $\sqrt[3]{0}$. The line $x = 0$, the tangent to the graph at 0, only meets the graph of $f(x)$ in one spot, and so our function will wind up passing the vertical line test (it's easy to see it passes the test for $x \neq 0$).

2. Homework questions - Spotting the Graph of a Derivative

We talked a bit about some homework questions, in particular the questions where you are given the graphs of 3 functions and then asked to figure out which graph is the derivative of another. To do this, we said

- You should start by selecting a graph and seeing where it has a tangent line with slope zero. Wherever your graph has a tangent line with slope 0, the graph of its derivative should pass through the x-axis.
- If there is more than one graph which passes through the x-axis in the correct places, you'll have multiple candidates for the graph of the derivative. Return to the graph of the function you're considering and see where slopes of tangents are positive or negative. You can then make sure the graph of the derivative takes on positive or negative values at these points.

3. More Derivative Shortcuts

In class last day we developed the following rules:

- $\frac{d}{dx}\left[f(x)+g(x)\right] = \frac{d}{dx}\left[f(x)\right] + \frac{d}{dx}\left[g(x)\right]$
- $\frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}\left[f(x)\right]$
- $\frac{d}{dx}\left[x^n\right] = nx^{n-1}$
- $\frac{d}{dx}\left[\sin(x)\right] = \cos(x).$

In fact, we can prove the first two of these by using the properties of the derivative. For instance

$$\begin{aligned}
\frac{d}{dx}\left[f(x)+g(x)\right] &= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-(f(x)+g(x))}{h} = \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right]\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= \frac{d}{dx}\left[f(x)\right]+\frac{d}{dx}\left[g(x)\right].
\end{aligned}$$

We could use a similar technique to prove the second rule. We proved the fourth rule in class Wednesday using lots of properties of $\sin(x)$ (it was bad, and not in a good way). We haven't yet seen how to prove the third rule (the so-called power rule), but we will in a couple of weeks.

Now I'm going to give you a few more derivative shortcuts. You need to memorize these!

$\frac{d}{dx}[e^x] = e^x$	$\frac{d}{dx}[\cos(x)] = -\sin(x)$
$\frac{d}{dx}[\tan(x)] = \sec^2(x)$	$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$
$\frac{d}{dx}[a^x] = (\ln(a))a^x$	

We can prove these rules using the definition of the derivative, but it's more important that you memorize these derivatives. Let's see that they are at least computable, though.

Example. Evaluate $\frac{d}{dx}[e^x]$ using $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$.

Solution. We'll evaluate the derivative using the definition, as per usual.

$$\frac{d}{dx}[e^x] = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x e^h - e^x}{h} = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h} = e^x.$$

The last line of our computation came from the fact about e^x we were given. □

A similar proof would show that $\frac{d}{dx}[a^x] = (\ln(a))a^x$, though you'd need to know that

$$\lim_{h\to 0} \frac{a^h - 1}{h} = \ln(a).$$