PRODUCT AND QUOTIENT RULES

1. Recap

We've entered the portion of the quarter where we start to become really proficient at computing derivatives. Today and tomorrow we're picking up some shortcuts that are going to make calculating derivatives easier.

2. The Product Rule

Last class period we saw that the derivative of a sum of functions is the sum of the derivative of each function. Is the same true for products? Let's investigate $\frac{d}{dx}[f \cdot g]$ with an example. Let's take $f(x) = g(x) = x$. Then we have

$$\frac{d}{dx}[f \cdot g] = \frac{d}{dx}\left[x^2\right] = 2x^{2-1} = 2x$$

and

$$\frac{d}{dx}[f]\frac{d}{dx}[g] = \frac{d}{dx}[x]\frac{d}{dx}[x] = 1 \cdot 1 = 1.$$

In this case, then, we have $\frac{d}{dx}[f \cdot g] \neq \frac{d}{dx}[f]\frac{d}{dx}[g]$. So what is the derivative of a product of functions?

The Product Rule. Suppose f and g are differentiable functions. Then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x).$$

Example. Evaluate $\frac{d}{dx}[xe^x]$.

Solution. With $f(x) = x$ and $g(x) = e^x$, we can evaluate this derivative using the product rule:

$$\frac{d}{dx}[xe^x] = x\frac{d}{dx}[e^x] + e^x\frac{d}{dx}[x] = xe^x + e^x = e^x(x+1).$$

□

Example. Evaluate $\frac{d}{dx}\left[(x^3 + x^2 + x + 1)(x^{17} - 3x^5 + 17x^2 + 1)\right]$.

Solution. Let's call $f(x) = x^3 + x^2 + x + 1$ and $g(x) = x^{17} - 3x^5 + 17x^2 + 1$. Then we know $f'(x) = 3x^2 + 2x + 1$ and $g'(x) = 17x^{16} - 3 \cdot 5x^4 + 17 \cdot 2x = 17x^{16} - 15x^4 + 34x$. Hence the product rule says

$$\frac{d}{dx}\left[(x^3 + x^2 + x + 1)(x^{17} - 3x^5 + 17x^2 + 1)\right] = f(x)g'(x) + g(x)f'(x)$$

$$= (x^3 + x^2 + x + 1)(17x^{16} - 15x^4 + 34x) + (x^{17} - 3x^5 + 17x^2 + 1)(3x^2 + 2x + 1).$$

We could expand this out and simplify, but there's no real need to. □

Example. Evaluate $\dfrac{d}{dx}\left[(x+1)(x^2-3)\right]$.

Solution. If we wanted to, we could use the product rule to compute this derivative. Indeed, let $f(x) = x+1$ and $g(x) = x^2 - 3$. Then $f'(x) = 1$ and $g'(x) = 2x$ (using the power rule), and so the product rule says

$$\frac{d}{dx}\left[(x+1)(x^2-3)\right] = f(x)g'(x) + g(x)f'(x) = (x+1)(2x) + (x^2-3)(1).$$

We could also evaluate this derivative by multiplying out $(x+1)(x^2-3) = x^3 - 3x + x^2 - 3$, and then taking the derivative of this polynomial. In this case we'd have

$$\frac{d}{dx}\left[(x+1)(x^2-3)\right] = \frac{d}{dx}\left[x^3 - 3x + x^2 - 3\right] = 3x^2 - 3 + 2x.$$

These two techniques will give you the same answer (after you simplify the first), but the second is slightly easier to do. □

Example. Evaluate $\dfrac{d}{dx}[f(x)g(x)h(x)]$.

Solution. This is a little tricky, because the product rule only tells us how to compute the derivative of a product of two functions. To use it in this case, we'll write $F(x) = f(x)g(x)$, so that we're actually evaluating $\dfrac{d}{dx}[F(x)h(x)]$. The product rule tells us this is

$$\frac{d}{dx}[F(x)h(x)] = F'(x)h(x) + F(x)h(x).$$

But what is $F'(x)$? Using the product rule again, we have

$$F'(x) = \frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x).$$

We can now plug this back into our first equation, and we'll have

$$\begin{aligned}\frac{d}{dx}[f(x)g(x)h(x)] = \frac{d}{dx}[F(x)h(x)] &= F'(x)h(x) + F(x)h(x) \\ &= (f'(x)g(x) + g'(x)f(x))h(x) + f(x)g(x)h'(x) \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).\end{aligned}$$

□

3. The Quotient Rule

It often happens that as you're walking down the street, someone stops you and asks you to compute a derivative. We're getting good at computing derivatives, but so far we still don't know how to compute something like $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$ in any easy way. We're going to figure out this derivative by doing something sneaky.

The Quotient Rule. Suppose that f and g are differentiable functions. Then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

Notice that in the formula above, it is VERY important that the function in the numerator is called $f(x)$ and the function in the denominator is called $g(x)$; the minus sign in our expression will give us problems if we switch these names around! So be careful!

Example. Evaluate $\frac{d}{dx}\left[\frac{\frac{2}{x}-e^x}{x^2+1}\right]$.

Solution. Using the quotient rule, we have

$$\begin{aligned}\frac{d}{dx}\left[\frac{\frac{2}{x}-e^x}{x^2+1}\right] &= \frac{d}{dx}\left[\frac{2x^{-1}-e^x}{x^2+1}\right]\\ &= \frac{(x^2+1)\frac{d}{dx}\left[2x^{-1}-e^x\right]-(2x^{-1}-e^x)\frac{d}{dx}\left[x^2+1\right]}{(x^2-1)^2}\\ &= \frac{(x^2+1)\left[-2x^{-2}-e^x\right]-(2x^{-1}-e^x)\left[2x\right]}{(x^2+1)^2}.\end{aligned}$$

We could simplify this answer if we wanted to, but we won't. □

Example. Evaluate $\frac{d}{dx}\left[\frac{x^3+x}{x^2}\right]$.

Solution. Write $f(x) = x^3 + x$ and $g(x) = x^2$. Then the quotient rule gives

$$\frac{d}{dx}\left[\frac{x^3+x}{x^2}\right] = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2} = \frac{x^2(3x^2+1)-(x^3+x)(2x)}{(x^2)^2}.$$

This is a perfectly acceptable answer, though one can make this calculation easier.

Indeed, our domination of fractional simplification tells us that $\frac{x^3+x}{x^2} = \frac{x^3}{x^2}+\frac{x}{x^2} = x+\frac{1}{x} = x+x^{-1}$. This means that

$$\frac{d}{dx}\left[\frac{x^3+x}{x^2}\right] = \frac{d}{dx}\left[x+x^{-1}\right] = 1+(-1)x^{-1-1} = 1-x^{-2} = 1-\frac{1}{x^2}.$$

□

Example. Compute $\frac{d}{dx}\left[\tan(x)\right]$.

Solution. We know from a previous lecture that $\frac{d}{dx}\left[\tan(x)\right] = \sec^2(x)$, but let's see this using the quotient rule.

First, we know that $\tan(x) = \frac{\sin(x)}{\cos(x)}$, and so $\frac{d}{dx}\left[\tan(x)\right]$ can be computed using the quotient rule with $f(x) = \sin(x)$ and $g(x) = \cos(x)$. Then we have

$$\frac{d}{dx}\left[\tan(x)\right] = \frac{d}{dx}\left[\frac{sin(x)}{\cos(x)}\right] = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$$

$$= \frac{\cos(x)(\cos(x))-\sin(x)(-\sin(x))}{\cos^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x).$$

Throughout I've assumed that $g'(x)$ and $f'(x)$ exist (so that that appropriate limits exist). I've used $f'(x)$ in a sneaky way: since $f'(x)$ exists, $f(x)$ has to be continuous, and so in the second to last line $\lim_{h\to 0} f(x+h) - f(x) = 0$. □

4. Some proofs

Proof of the Product Rule. What is $\frac{d}{dx}[f \cdot g]$? We'll work from the definition:

$$\begin{aligned}
\frac{d}{dx}[f \cdot g] &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \frac{\left[f(x) + \overbrace{f(x+h) - f(x)}^{F}\right]\left[g(x) + \overbrace{g(x+h) - g(x)}^{G}\right] - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x)g(x) + f(x)G + g(x)F + FG - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x)G}{h} + \lim_{h \to 0} \frac{g(x)G}{h} + \lim_{h \to 0} \frac{FG}{h} \\
&= f(x) \underbrace{\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}}_{g'(x)} + g(x) \underbrace{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}_{f'(x)} \\
&+ \underbrace{\lim_{h \to 0} f(x+h) - f(x)}_{0} \underbrace{\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}}_{g'(x)} \\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

□

Proof of the Quotient Rule. What is $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$? I'll compute it by turning this into a problem where I can use the product rule.

First, I set up a dummy variable by setting $F = \frac{f}{g}$. This gives $Fg = f$, an equality I can use to my advantage by taking the derivative of each side and setting them equal to each other. We find $f'(x) = F(x)g'(x) + F'(x)g(x)$ by the product rule, and solving for $F'(x)$ we find

$$\begin{aligned}
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = F'(x) = \frac{f'(x) - F(x)g'(x)}{g(x)} &= \frac{\frac{g(x)}{g(x)}f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)} \\
&= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}
\end{aligned}$$

□