THE CHAIN RULE

1. Recap

We've been focusing lately on developing tricks for computing derivatives of complicated functions. Last class period we learned the product and chain rules, which allow us to compute the derivative of a product or quotient of functions in terms of the constituent functions and their derivatives. Just as a reminder, let's do a derivative that will require both of these tools.

Example. Compute $\frac{d}{dx}\left[\frac{xe^x}{x^2+1}\right]$.

Solution. We'll write $f(x) = xe^x$ and $g(x) = x^2 + 1$. The product rule gives $f'(x) = xe^x + e^x$, and the power rule gives $g'(x) = 2x$. The quotient rule then gives us

$$\frac{d}{dx}\left[\frac{xe^x}{x^2+1}\right] = \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} = \frac{(xe^x + e^x)(x^2+1) - 2x(xe^x)}{(x^2+1)^2}.$$

□

Today we're going to learn how to evaluate the derivative of a composition of functions in much the same way.

2. The Chain Rule

The Chain Rule. Suppose $f(x)$ and $g(x)$ are differentiable functions. Then

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))g'(x).$$

The chain rule is an unbelievably powerful tool, and together with the product and quotient rules it will let us evaluate the derivative of just about any function we could ever come across. It is generally met with some difficulty when encountered for the first time, but after you do several computations with the chain rule you should find it very usable. Today we'll just do lots of examples, and in your homework you'll do several more examples.

Example. Compute $\frac{d}{dx}\left[\sqrt{1+2x}\right]$.

Solution. The first thing to notice when you're doing a derivative like this is that this is, first and foremost, a composition of functions. The 'outside' function is $\sqrt{x}$, and the 'inside' function is $1 + 2x$; in other words, with $f(x) = \sqrt{x}$ and $g(x) = 1 + 2x$, we have $\sqrt{1+2x} = f(g(x))$. Hence, to evaluate the derivative in question, we need to use the chain rule: $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))g'(x)$.

We have seen many times over that $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ (using the power rule), and similarly we know $g'(x) = 2$ (again, using the power rule). Using these computations, we have

$$\frac{d}{dx}\left[\sqrt{1+2x}\right] = f'(g(x)g'(x) = \left(\frac{1}{2\sqrt{x}} \circ 1 + 2x\right)(2) = \frac{2}{2\sqrt{1+2x}} = \frac{1}{\sqrt{1+2x}}.$$

□

Example. Compute $\frac{d}{dx}\left[\sqrt{1+x^2}\right]$

Solution. This functions is amazingly similar to the last one. Again we can write out function as a composition of functions$\sqrt{1+x^2} = f(g(x))$, and again the outside function is $f(x) =\sqrt{x}$. The difference is the inner function, $g(x) = 1 + x^2$. To compute $\frac{d}{dx}\left[\sqrt{1+x^2}\right]$, then, we use the chain rule. This means we need to know $f'(x)$ and $g'(x)$: the first is $f'(x) = \frac{1}{2\sqrt{x}}$ as before, and $g'(x) = 2x$ from the power rule.

The chain rule then gives

$$\frac{d}{dx}\left[\sqrt{1+x^2}\right] = \frac{1}{2\sqrt{1+x^2}}(2x) = \frac{2x}{2\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}.$$

□

Example. Compute $\frac{d}{dx}[\sin(2x)]$.

Solution. The first thing to recognize is that $\sin(2x)$ is the composition $f(g(x))$, where $f(x) = \sin(x)$ and $g(x) = 2x$. Hence to compute $\frac{d}{dx}[\sin(2x)]$ we use the chain rule: $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$. In this case $f'(x) = \cos(x)$ and $g'(x) = 2$, and so

$$\frac{d}{dx}[f(g(x))] = \cos(2x)(2) = 2\cos(2x).$$

□

Example. Compute $\frac{d}{dx}[\sin(\cos(x))]$

Solution. Again, we begin by noticing $\sin(\cos(x)) = f(g(x))$, where $f(x) = \sin(x)$ and $g(x) = \cos(x)$. Hence the derivative in question can be calculated using the chain rule, for which we'll need to know $f'(x) = \cos(x)$ and $g'(x) = -sin(x)$. We then have

$$\frac{d}{dx}[\sin(\cos(x))] = \frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = \cos(\cos(x))(-\sin(x)) = -\sin(x)\cos(\cos(x)).$$

□

These last four examples give a good illustration of how the chain rule takes derivatives of seemingly unrelated problems and draws connections between them. For instance, the last two examples concerned a relatively innocuous function $\sin(2x)$ and a totally bizarre function $\sin(\cos(x))$. Whereas they have very different derivatives, computing the derivatives in each case required essentially the same amount of work. The chain rule is the jackhammer in our toolbox!

Example. Compute $\frac{d}{dx}\left[e^{x^2+1}\right]$.

Solution. You won't be surprised to see that we compute this derivative using the chain rule, since $e^{x^2+1} = f(g(x))$ for $f(x) = e^x$ and $g(x) = x^2 + 1$. Since $f'(x) = e^x$ and $g'(x) = 2x$, the chain rule gives

$$\frac{d}{dx}\left[e^{x^2+1}\right] = \frac{d}{dx}\left[f(g(x))\right] = f'(g(x))g'(x) = e^{x^2+1}(2x) = 2xe^{x^2+1}.$$

□

All the examples we have done so far have used the chain rule to compute the derivative of a composition of two relatively innocuous functions. What if, instead, we want the derivative of the composition of three functions? More specifically, what is $\frac{d}{dx}\left[f(g(h(x)))\right]$? Write $G(x) = g(h(x))$, and notice the chain rule gives $G'(x) = g'(h(x))h'(x)$. Now we have

$$\frac{d}{dx}\left[f(g(h(x)))\right] = \frac{d}{dx}\left[f(G(x))\right] = f'(G(x))G'(x) = f'(g(h(x))\ g'(h(x))\ h(x).$$

We could do a similar trick to show

$$\frac{d}{dx}\left[f(g(h(i(x))))\right] = f'(g(h(i(x))))\ g'(h(i(x)))\ h'(i(x))\ i'(x),$$

or even more complicated compositions toward the same pattern. In fact, it is exactly these computations that gave the chain rule it's name. We will certainly be evaluating the derivative of a composition of three functions, so you should remember this trick so you don't have to re-derive it yourself on a test or quiz!

Example. Compute $\frac{d}{dx}\left[e^{\sqrt{x^2+1}}\right]$.

Solution. This time we notice that our function is the composition $f(g(h(x)))$, where $f(x) = e^x$, $g(x) = \sqrt{x}$, and $h(x) = x^2 + 1$. From our discussion above, we know that $\frac{d}{dx}\left[f(g(h(x)))\right] = f'(g(h(x)))\ g'(h(x))\ h'(x)$, so we need to compute $f'(x) = e^x$, $g'(x) = \frac{1}{2\sqrt{x}}$, and $h'(x) = 2x$. Then we have

$$\frac{d}{dx}\left[e^{\sqrt{x^2+1}}\right] = \frac{d}{dx}\left[f(g(h(x)))\right] = f'(g(h(x)))\ g'(h(x))\ h'(x) = e^{\sqrt{x^2+1}}\ \frac{1}{2\sqrt{x^2+1}}\ 2x = \frac{x}{\sqrt{x^2+1}}e^{\sqrt{x^2+1}}.$$

□

Example. Compute $\frac{d}{dx}\left[\sqrt{x+\tan(\sqrt{x})}\right]$.

Solution. We start by recognizing that this function is a, first and foremost, a composition $f(g(x))$, where $f(x) = \sqrt{x}$ and $g(x) = x + \tan(\sqrt{x})$. *NB. You might want this to be a composition of three functions, like the last problem, but you'll see that it's quite difficult to write $g(x)$ as a composition of two functions; the + really screws things up.* Hence to compute the desired derivative, we need to know $f'(x)$ and $g'(x)$. The first is pretty easy using the power rule: $f'(x) = \frac{1}{2\sqrt{x}}$. The second, however, requires some work.

We know that $\frac{d}{dx}[x] = 1$, but we don't know $\frac{d}{dx}[\tan(\sqrt{x})]$. To find this, we'll need to use the chain rule. Indeed, $\tan(\sqrt{x}) = h(i(x))$ where $h(x) = \tan(x)$ and $i(x) = \sqrt{x}$. The chain gives

$$\frac{d}{dx}\left[\tan(\sqrt{x})\right] = h'(i(x))\ i'(x) = \sec^2(\sqrt{x})\ \frac{1}{2\sqrt{x}} = \frac{\sec^2(\sqrt{x})}{2\sqrt{x}}$$

and so

$$g'(x) = \frac{d}{dx}\left[x + \tan(\sqrt{x})\right] = 1 + \frac{\sec^2(\sqrt{x})}{2\sqrt{x}}.$$

Now that we know both $f'(x)$ and $g'(x)$, we can compute

$$\frac{d}{dx}\left[\sqrt{x + \tan(\sqrt{x})}\right] = \frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\ g'(x) = \frac{1}{2\sqrt{x + \tan(\sqrt{x})}}\ \left(1 + \frac{\sec^2(\sqrt{x})}{2\sqrt{x}}\right).$$

$\square$