PRACTICE CALCULATING DERIVATIVES

1. Homework problems

Class started by doing some computations for the homework assignment. The solutions to the homework have been posted, so you can check there for answers to some of these questions. If you get stuck doing one of these problems and want to see where you went wrong, feel free to email me so we can sort it out.

2. Recap

We started this class talking about the tangent problem: given the graph $y = f(x)$ and a point $(a, f(a))$ on the graph, find the slope of the line which is tangent to $y = f(x)$ at the point $(a, f(a))$. To solve this problem we said we needed to understand expressions of the form

$$\lim_{x_1 \to x_2} \frac{f(x_1) - f(x_2)}{x_1 - x_2},$$

and so we first had to make precise the intuitive notion of limits. Once we had done this we could calculate the derivative of a function: $f'(a)$ is the slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$, and is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

One we had solved the tangent problem we started considering the derivative of $f(x)$ as a function itself: for an arbitrary input x, $f'(x)$ gives the slope of the tangent line to f at $(x, f(x))$. When we first started working with this, we only had the definition of the derivative at our disposal to calculate derivatives. We saw that this could make computing the derivative of some simple functions (like $f(x) = x^3$) tedious, while it made calculating the derivatives of other functions (like $f(x) = \sqrt[3]{x}$) super hard.

In the last week we have written down lots of rules that make calculating derivatives *much* simpler. We started with the power rule (which lets us compute derivatives of polynomials) together with derivative rules for trig functions and exponentials. We then discussed the product and quotient rules, which made it possible to compute the derivative of a product or quotient by knowing the derivatives of the factors. Yesterday in class we talked about the chain rule, which let's us compute the derivative of a composition of functions in terms of the constituent functions. With these tools at hand, we can now compute just about any derivative you'll ever want to see. For instance, if we were so inclined, we could now compute the derivative of

$$x^2 e^{\sqrt{x^3 + \sin(\tan(x^2+1))}}.$$

Today in class we're going to step back and compute a lot of derivatives using the new tools at our disposal.

3. Computing derivatives of inverse functions

The chain rule gives us a nifty way to compute the derivative of the inverse of a function. Let's see it in action a few times.

Example. Compute $\frac{d}{dx}[\ln(x)]$.

Solution. We're going to compute $\frac{d}{dx}[\ln(x)]$ in a tricky way.

First, we remember that $\ln(x)$ has the property that

$$e^{\ln(x)} = x.$$

Now let's compute the derivative of each side. On the right hand side we have $\frac{d}{dx}[x] = 1$. What about the left hand side? We compute this using the chain rule (with $f(x) = e^x$ and $g(x) = \ln(x)$). The chain rule says that $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$, and so we have

$$\frac{d}{dx}\left[e^{\ln(x)}\right] = e^{\ln(x)}\frac{d}{dx}[\ln(x)].$$

Setting these two derivatives equal to each other, we have $e^{\ln(x)}\frac{d}{dx}[\ln(x)] = 1$. Since $e^{\ln(x)} = x$, we can divide by x on both sides to obtain

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}.$$

□

Example. Compute $\frac{d}{dx}[\arcsin(x)]$.

Solution. Remember that $\arcsin(x)$ satisfies the equation $\sin(\arcsin(x)) = x$. Now let's compute the derivative of each side. On the right we get derivative 1, and on the left we use the chain rule (with $f(x) = \sin(x)$ and $g(x) = \arcsin(x)$) to find

$$\frac{d}{dx}[\sin(\arcsin(x))] = \frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = \cos(\arcsin(x))\frac{d}{dx}[\arcsin(x)].$$

Now setting our derivatives equal to each other and dividing through by $\cos(\arcsin(x))$, we see

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\cos(\arcsin(x))}.$$

If one were so inclined, you could simplify this even further, and you would find

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}.$$

We'll do this next week. □

After these two computations you can see we're really just exploiting a common principle. One could use this same trick to compute the derivatives of the inverses of all kinds of functions. In fact, one of the problems on the worksheet is to use this trick to compute $\frac{d}{dx}[\sqrt[3]{x}]$.