IMPLICIT DIFFERENTIATION

1. Recap

We've recently become amazingly good at computing derivatives. The tools we've used to develop this proficiency are basic derivatives of polynomials, trigonometic functions, exponentials, and certain 'basic' algebraic functions. With these in hand we use the product, quotient, and chain rules to compute any derivative that comes our way.

Last class period we saw how to use the chain rule to compute the derivative of the inverse of a function. In particular, we saw that

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x} \quad \text{and} \quad \frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}.$$

Today we'll apply our new muscles to solve the tangent problem for graphs that aren't even functions!

2. The tangent problem for the circle

One of the most basic objects in mathematics is the circle. Analytically, a circle of radius 1 centered at the origin is represented by the formula $x^2 + y^2 = 1$. As it is one of the fundamental mathematical graphs, it is natural to want to solve the tangent problem for the circle. Sadly, however, the graph of the circle is not the graph of a function because it fails the vertical line test. Since we only have tools for solving the tangent problem for functions, we can't yet solve the tangent problem for the circle.

How can a person remedy this problem? One trick would be to take the expression for the circle and solve for y. This would give an expression for y in terms of x, an expression which we might then be able to evaluate the derivative of. In this case, solving for y gives $y = \pm\sqrt{1-x^2}$. The $+$ and $-$ are there because we've split the circle into a top piece $y = \sqrt{1-x^2}$ and a bottom piece $y = -\sqrt{1-x^2}$. We could then evaluate the derivative of each of these functions and then use them to find the slope of the line tangent to a point (x, y) on the circle (which derivative we used would depend on which half of the circle (x, y) lived on). This would result in derivatives

$$-\frac{x}{\sqrt{1-x^2}} \quad \text{and} \quad \frac{x}{\sqrt{1-x^2}}.$$

There are a few problems with this approach.

- From an aesthetic standpoint, it's pretty clunky and unnatural. Given an expression that's nice like $x^2 + y^2 = 1$, it's silly that we need to evaluate ugly derivatives like $\sqrt{1-x^2}$.
- From a practical standpoint, it's problematic because to find the slope of the tangent to the curve at a point P we have to figure out which of the two functions P 'lives on.' This isn't too much of a problem for the circle, but can become more complicated when we have nastier expressions.
- From a computability standpoint, given a complicated expression involving x's and y's, there's no guarantee that we'll even be able to solve for y! Nevertheless, the tangent problem still makes perfect sense for these complicated graphs, so we need a new tool.

3. Implicit Differentiation

To solve these problems, we develop a new way to evaluate derivatives of *implicit* functions of x.

Example. Find the slope of the line tangent to the graph $x^2+y^2=1$ at an arbitrary point (x,y) on the curve.

Solution. We're going to use a new tool called implicit differentiation to solve this problem. Our idea will be to take the given expression $x^2+y^2=1$ and evaluate the derivative of both the left and right hand sides. In doing this, the desired derivative (which is $\frac{dy}{dx}$, since this represents 'rise over run') will pop out, and we'll be able to solve.

The derivative of the right hand side of our expression is easy: $\frac{d}{dx}[1]=0$. The left hand side is slightly more complicated: $\frac{d}{dx}\left[x^2+y^2\right]=2x+2y\frac{dy}{dx}$. Why does this strange factor of $\frac{dy}{dx}$ show up? Essentially, this is just the chain rule. In this case the variable y is *implicitly* a function of x, and so when we evaluate its derivative we need to use the chain rule. To see how it appears, let's write $y=f(x)$. Then

$$\frac{d}{dx}\left[y^2\right]=\frac{d}{dx}\left[(f(x))^2\right]=2f(x)f'(x)=2y\frac{dy}{dx}.$$

Hence we have the stated equality: $\frac{d}{dx}\left[x^2+y^2\right]=2x+2y\frac{dy}{dx}$.

Setting the derivatives of the left and right hand sides equal gives $2x+2y\frac{dy}{dx}=0$. Now since we're after $\frac{dy}{dx}$, we can just solve:

$$\frac{dy}{dx}=\frac{-2x}{2y}=-\frac{x}{y}.$$

This means that the slope of the line tangent to the graph $x^2+y^2=1$ at a point (x,y) is given by $-\frac{x}{y}$. As a reality check, we can verify this formula against what we know the slopes of the tangent to the circle at certain 'nice' points to be. $\square$

This examples embodies everything there is in implicit differentiation. To solve the tangent problem for a graph which isn't explictly a function of x (ie, in an expression which isn't 'solved for y'), the technique is pretty simple:

- start with a complicated expression involving x's and y's; this makes y an implicit function of x;
- compute the derivative of both sides of this expression (don't forget those $\frac{dy}{dx}$'s which pop up!);
- solve for $\frac{dy}{dx}$;
- celebrate, because $\frac{dy}{dx}$ gives you the slope of the line tangent to the curve defined by your original complicated expression.

Let's do some more examples.

Example. Solve the tangent problem for the point $(1,1)$ on the curve $y^2 = x^3(2-x)$.

Solution. We want to solve for $\frac{dy}{dx}$, so we need to compute the derivative of the left and right hand sides of the given equality. The right hand side has derivative $\frac{d}{dx}\left[x^3(2-x)\right] = \frac{d}{dx}\left[2x^3 - x^4\right] = 6x^2 - 4x^3$, and the left hand side has derivative $\frac{d}{dx}\left[y^2\right] = 2y\frac{dy}{dx}$ (remember: the $\frac{dy}{dx}$ is appearing because of our old friend the chain rule). Hence we have

$$2y\frac{dy}{dx} = 6x^2 - 4x^3.$$

Solving for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = \frac{6x^2 - 4x^3}{2y} = \frac{3x^2 - 2x^3}{y}.$$

This means the slope of the line tangent to the curve at $(1,1)$ is $\frac{3(1)^2 - 2(1)^3}{1} = \frac{1}{1} = 1$. Hence the line tangent to the curve at $(1,1)$ is

$$y - 1 = 1(x-1) \quad \text{or, equivalently } y = x.$$

□

Example. Compute $\frac{d}{dx}[\arcsin(x)]$.

Solution. This doesn't look like an implicit differentiation problem, but we're going to make it one. We start by writing $y = \arcsin(x)$. Applying sine to both sides then gives $\sin(y) = \sin(\arcsin(x)) = x$. Aha! Now we can use implicit differentiation. The left hand side has derivative $\cos(y)\frac{dy}{dx}$, and the right hand side has derivative 1. This means we have

$$\frac{dy}{dx} = \frac{1}{\cos(y)}.$$

But what is $\cos(y)$ in terms of x? We know that $\cos^2(y) + \sin^2(y) = 1$, so that $\cos(y) = \sqrt{1 - \sin^2(y)}$. But since $\sin(y) = x$, we have $\cos(y) = \sqrt{1-x^2}$. Hence, we have

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-x^2}}.$$

□

Example. Compute $\frac{d}{dx}[\arccos(x)]$.

Solution. We'll use the same trick as last time. We start by writing $y = \arccos(x)$ and apply cosine to both sides to get $\cos(y) = \cos(\arccos(x)) = x$. The left hand side has derivative $-\sin(y)\frac{dy}{dx}$, and the right hand side has derivative 1. This means we have

$$\frac{dy}{dx} = -\frac{1}{\sin(y)}.$$

But what is $\sin(y)$ in terms of x? Again, we have $\cos^2(y) + \sin^2(y) = 1$, so $\sin(y) = \sqrt{1-\cos^2(y)}$. But since $\cos(y) = x$, we have $\sin(y) = \sqrt{1-x^2}$. Hence, we have

$$\frac{d}{dx}[\arcsin(x)] = -\frac{1}{\sin(\arccos(x))} = -\frac{1}{\sqrt{1-x^2}}.$$

□

Example. Show that $\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}$.

Solution. We use the same trick as before, but when simplifying this we'll need the identity $1+\tan^2(y) = \sec^2(y)$. Try it! □

Example. Find $\frac{dy}{dx}$ for the graph $\sqrt{1+x^2y^2} = 2xy$.

Solution. In this problem I'm going to use the shorthand $y' = \frac{dy}{dx}$. A lot of students find this useful, and I want you to feel comfortable using it if you like.

Ok, this is a typical implicit differentiation problem. We begin by computing the derivatives of the left and right hand sides. For the right hand side, we have

$$\frac{d}{dx}[2xy] = 2(xy' + y) = 2xy' + 2y$$

(note: I had to use the product rule to compute this derivative). For the left hand side, I notice that $\sqrt{1+x^2y^2} = f(g(x))$, where $f(x) = \sqrt{x}$ and $g(x) = 1 + x^2y^2$. Hence the chain rule says that

$$\frac{d}{dx}\left[\sqrt{1+x^2y^2}\right] = \frac{1}{2\sqrt{1+x^2y^2}}\left(x^2 \cdot 2yy' + 2xy^2\right) = \frac{2x^2yy'}{2\sqrt{1+x^2y^2}} + \frac{2xy^2}{2\sqrt{1+x^2y^2}}.$$

Setting these two derivatives equal to each other, I solve for y':

$$2xy' + 2y = \frac{2x^2yy'}{2\sqrt{1+x^2y^2}} + \frac{2xy^2}{2\sqrt{1+x^2y^2}}$$

$$\iff \quad 2y - \frac{2xy^2}{2\sqrt{1+x^2y^2}} = \frac{2x^2yy'}{2\sqrt{1+x^2y^2}} - 2xy'$$

$$\iff \quad 2y - \frac{2xy^2}{2\sqrt{1+x^2y^2}} = y'\left(\frac{2x^2y}{2\sqrt{1+x^2y^2}} - 2x\right)$$

$$\iff \quad y' = \frac{2y - \frac{2xy^2}{2\sqrt{1+x^2y^2}}}{\frac{2x^2y}{2\sqrt{1+x^2y^2}} - 2x}$$

The answer isn't pretty, but that's ok. □