THE CALCULUS/GEOMETRY DICTIONARY

1. Announcements

We have a test coming up next Friday. This means

- I'll post a practice midterm sometime later today (Friday) or possibly tomorrow. The practice midterm should be similar in length and scope to your actual midterm. My strong suggestion is that you treat the practice midterm as an actual midterm, carving out 50 minutes from your schedule to take the test without interruption.
- Your next homework assignment has been posted and is due next Wednesday. It covers the material we'll talk about in class today and Monday.
- We'll have a review session next Thursday, tentatively starting around 6:15 or 6:30. Like last time, I'll bring pizza. You bring questions.

2. Recap

In the last few weeks we've gotten really great at computing derivatives of functions. The product, quotient, and chain rules have opened up the door to evaluating derivatives of even the most complicated functions, and implicit differentiation let's us find slopes of tangents to graphs that aren't even functions. Truly, we have become wildly powerful.

Sadly, however, we don't have a lot of applications for derivatives. We wouldn't be spending a whole term on derivatives if they only solved the tangent problem. In fact, derivatives have lots of interesting applications. We'll spend the next week talking about how one can use calculus to analyze the graph of a function in some fairly sophisticated ways.

3. A bevy of new terms

Before I introduce the dictionary between calculus and geometry, I need to define a few terms for you.

Definition. A function $f(x)$ is increasing if $f(x_0) \geq f(x_1)$ whenever $x_0 > x_1$. It is strictly increasing if $f(x_0) > f(x_1)$ whenever $x_0 > x_1$. A function $f(x)$ is decreasing if $f(x_0) \leq f(x_1)$ whenever $x_0 > x_1$. It is strictly decreasing if $f(x_0) < f(x_1)$ whenever $x_0 > x_1$.

Intuitively, a function is increasing if outputs don't get smaller as inputs get bigger. Similarly, a function is decreasing if outputs don't get bigger as inputs get bigger.

Definition. A function $f(x)$ is concave up if its derivative is strictly increasing. A function $f(x)$ is concave down if its derivative is strictly decreasing.

Intuitively, a function is concave up if the graph of $f(x)$ 'holds water.' You can imagine the corresponding statement for functions which are concave down.

Definition. A function has an inflection point at a if the concavity of $f(x)$ changes at a.

Definition. Critical points of $f(x)$ are values where $f'(x) = 0$ or where $f'(x)$ is undefined.

You've already done homework problems that teach you how to identify critical points of a function based on the graph of $f(x)$, though soon we'll also be using analytic expressions to compute critical points as well.

Definition. A function $f(x)$ has a local maximum at a if, whenever b is a point 'near' a, $f(b) \leq f(a)$. A function $f(x)$ has a local minimum at a if, whenever b is a point 'near' a, $f(b) \geq f(a)$.

Graphically, local maxima and minima are places where the graph of $f(x)$ peaks. There are a few problems with the definitions I've given you, most notably that it is hard to know exactly what it means for a point b to be 'near' a. We'll talk more about this on class on Monday, and even find a way to resolve it.

Finally, I'll frequently want to refer to the collection of local maxima and local minima of a function, so it's convenient to have a name for this set.

Definition. Local extrema of a function $f(x)$ are local maxima and local minima of $f(x)$.

4. The Calculus/Geometry Dictionary

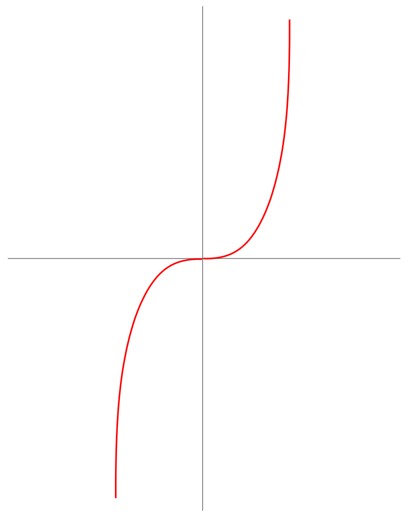
Information about $f'(x)$ and $f''(x)$ can be used to cull information about the graph of $f(x)$. The correspondence between derivative information and graphical information is what I call the calculus/geometry dictionary. As the name suggests, the dictionary allows you to translate information about the calculus of a function (its derivatives) into information about the geometry of a function (its graph) and vice versa. It's a very powerful tool that you'll be using for the rest of your calculus life. The following list is by no means exhaustive, but covers most of the things we'll be interested in.

$f(x)$ is flat at a	$\leftrightarrow$	$f'(a) = 0$
$f(x)$ is increasing on (a, b)	$\leftrightarrow$	$f'(x) \geq 0$ on (a, b)
$f(x)$ is decreasing on (a, b)	$\leftrightarrow$	$f'(x) \leq 0$ on (a, b)
$f(x)$ is concave up on (a, b)	$\leftrightarrow$	$f''(x) > 0$ on (a, b)
$f(x)$ is concave down on (a, b)	$\leftrightarrow$	$f''(x) < 0$ on (a, b)
$f'(x)$ is increasing on (a, b)	$\leftrightarrow$	$f''(x) > 0$ on (a, b)
$f'(x)$ is decreasing on (a, b)	$\leftrightarrow$	$f''(x) < 0$ on (a, b)
Slopes of tangent lines of f are increasing on (a, b)	$\leftrightarrow$	$f''(x) \geq 0$ on (a, b)
Slopes of tangent lines of f are decreasing on (a, b)	$\leftrightarrow$	$f''(x) \leq 0$ on (a, b)

We'll normally be using the dictionary to translate derivative information into graphical information, but let's do a few examples where we convert graphical information into derivative information.

Example. Consider the graph of $f(x) = x^3$. Use it to find critical points, intervals on which $f'(x) > 0$, intervals on which $f'(x) < 0$, intervals on which $f''(x) > 0$, intervals on which $f''(x) < 0$, and inflection points.

Solution. The graph of $f(x) = x^3$ looks something like:



From this graph we see that graph has a critical point at $x = 0$, since at this point the graph 'flattens out' and so has derivative 0. There are no other places where the tangent has slope 0 or where the derivative is undefined, and so $x = 0$ is the only critical point.

We also see that everywhere except 0, the slopes of tangent lines are positive. This means that $f'(x) > 0$ everywhere except 0, i.e. on the set $(-\infty, 0) \cup (0, \infty)$. According to the dictionary, this means $f'(x) > 0$ on the intervals $(-\infty, 0)$ and $(0, \infty)$. This also means that there are no intervals on which $f'(x) < 0$.

Finally, we see that to the left of $x = 0$ the graph is concave down, whereas to the right of $x = 0$ the graph is concave up. According to our dictionary, this means $f''(x) < 0$ on $(-\infty, 0)$ and $f''(x) > 0$ on $(0, \infty)$. Since concavity changes at 0, this means $x = 0$ is an inflection point. □

Example. For the function $f(x) = -\cos(x)$, find critical points, intervals on which $f'(x) > 0$, intervals on which $f'(x) < 0$, intervals on which $f''(x) > 0$, intervals on which $f''(x) < 0$, and inflection points on the interval $[-\pi, \pi]$ using the graph of $f(x)$.

Solution. Draw the graph of $-\cos(x)$ on the interval $[-\pi, \pi]$ to follow along.

The first thing you notice is that the derivative $f'(x)$ is defined everywhere and is 0 at $-\pi, 0$, and π. These are the critical point on the interval.

Once you have the critical points, you'll notice that the sign of the derivative is constant on intervals between the critical points. Indeed, you can see the function is decreasing on $(-\pi, 0)$ and increasing on $(0, \pi)$. By the calculus/geometry dictionary, this means $f'(x) < 0$ on $(-\pi, 0)$ and $f'(x) > 0$ on $(0, \pi)$.

You can also see by inspection that the function has inflection points at $x = \pm\frac{\pi}{2}$, being concave down on $(-\pi, -\frac{\pi}{2})$ and $(\frac{\pi}{2}, \pi)$ and concave up on $(-\frac{\pi}{2}, \frac{\pi}{2})$. □