PRACTICE WITH APPLICATIONS OF DERIVATIVES

1. Announcements

Don't forget that we have a test on Friday! Because of the test,

- We have a review session Thursday night from 6:30 on. It will be in room 383-N like last time. Again like last time, I'll bring pizza and sodas.
- Like last time, the exam will begin exactly at 9. If you want to read the exam before you take it, be in the room at 8:55.
- I'll post solutions to the practice midterm today. I'll also have solutions to Homework 7 up.

2. Analyzing functions with calculus

Today I just want to review the big ideas for how we can use calculus to analyze the graph of a function. We started by people asking questions about homework questions. I realize now that the big problem we worked on was actually different from your homework problem. I've given a complete solution to the homework problem in the solution to Homework 7, but I'll include the discussion we had in class today because it's a nice example nonetheless.

Example. Let $f(x) = xe^{-x^2}$. Find intervals on which $f(x)$ is increasing/decreasing. While you're at it, find the local extrema of f. Also find intervals on which $f(x)$ is concave up/down together with inflection points.

NB: The homework asked you the following question: Suppose $f'(x) = xe^{x^2}$. Find intervals on which $f(x)$ is increasing/decreasing together with local extrema. Notice that this is different then that question above!

Solution. Because of the calculus/geometry dictionary, we know that to answer these questions we'll need to be able to analyze $f'(x)$ and $f''(x)$. For this, of course, we need to compute these derivatives. The product rule and chain rule come into play when evaluating:

$$f'(x) = \frac{d}{dx}\left[xe^{-x^2}\right] x(-2xe^{-x^2}) + e^{-x^2} = e^{-x^2}(1-2x^2)$$

$$f''(x) = \frac{d}{dx}\left[e^{-x^2}(1-2x^2)\right] = -2xe^{-x^2}(1-2x^2) + e^{-x^2}(-4x) = e^{-x^2}(-2x+4x^3-4x) = e^{-x^2}(x)(4x^2-6)$$

Alright, with this information I can ask: where are the critical points of $f(x)$? Since $e^{-x^2} > 0$ everywhere, this means $f'(x) = 0$ if and only if $1 - 2x^2 = 0$. But you can see that this occurs exactly at the points $\pm\frac{1}{\sqrt{2}}$. Since $f'(x)$ is defined everywhere, that means these two points are the only critical points.

With this in mind, I can sample the derivatives along the intervals $(-\infty, -frac1\sqrt{2})$, $(-\frac{1}{\sqrt{2}}), \frac{1}{\sqrt{2}})$, and $(\frac{1}{\sqrt{2}}, \infty)$ to determine where $f'(x)$ is positive or negative. Choosing -5 for the first interval, 0 for the second, and 5 for the last, I plug these values into $f'(x)$ and see that

- $f'(x) > 0$ on the interval $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, so that f is increasing on this interval; and that
- $f'(x) < 0$ on the intervals $(-\infty, -frac1\sqrt{2})$ and $(\frac{1}{\sqrt{2}}, \infty)$, so that f is decreasing on these intervals.

I can also use this information to perform the first derivative test. Since slopes go from negative to positive at $x = -\frac{1}{\sqrt{2}}$, we know that f has a local minimum at $x = -\frac{1}{\sqrt{2}}$. Since slopes go from positive to negative at $x = \frac{1}{\sqrt{2}}$, $f(x)$ has a local maximum there.

What about concavity and inflection? First I'll be interested in points where $f''(x) = 0$ or is undefined. Given that $e^{-x^2} > 0$ everywhere, we see that $f''(x) = 0$ exactly when $x(4x^2 - 6) = 0$. But this occurs at points where either $x = 0$ or $4x^2 - 6 = 0$, which together are the points $x = 0, \pm\sqrt{\frac{6}{4}}$.

Given these critical points of $f'(x)$, we can sample the sign of $f''(x)$ on the intervals

$$\left(-\infty, -\sqrt{\frac{6}{4}}\right), \left(-\sqrt{\frac{6}{4}}, 0\right), \left(0, \sqrt{\frac{6}{4}}\right) \text{ and } \left(\sqrt{\frac{6}{4}}, \infty\right)$$

by choosing points within these intervals and then determining the sign of $f''(x)$ at these sampled points. Choosing the points $-10, -1, 1$ and 10 for my sample points, I find that

- $f''(x) > 0$ on the intervals $(-\sqrt{\frac{6}{4}}, 0)$ and $(\sqrt{\frac{6}{4}}, \infty)$, so that $f(x)$ is concave up on these intervals.
- $f''(x) < 0$ on the intervals $(-\infty, -\sqrt{\frac{6}{4}})$ and $(0, \sqrt{\frac{6}{4}})$, so that $f(x)$ is concave down on these intervals.

With this information, I can also see that concavity changes at each of the points $x = 0$, $x = -\sqrt{\frac{6}{4}}$ and $x = \sqrt{\frac{6}{4}}$, so that these are the points of inflection.

If we wanted to, we could sketch the graph of this function using this information. In fact, we did just this in class. I won't put it here in the notes, though. If you'd like to see me do it again, don't be bashful about asking about it in office hours or in an email. □

Example. Let $f(x) = \dfrac{e^x}{1 + e^x}$. Find intervals on which $f(x)$ is increasing/decreasing along with local extrema of f. Also find intervals on which $f(x)$ is concave up/down together with inflection points.

Solution. To do this analysis, we know that we'll need to know $f'(x)$ and $f''(x)$. These require the quotient rule, and the second derivative will also require the use of the chain rule. We proceed to evaluate

$$f'(x) = \frac{(1 + e^x)e^x - e^x \ e^x}{(1 + e^x)^2} = \frac{e^x}{(1 + e^x)^2}$$

$$f''(x) = \frac{(1 + e^x)^2 e^x - e^x(2(1 + e^x)(e^x))}{(1 + e^x)^4} = \frac{e^x(1 - e^{2x})}{(1 + e^x)^4}$$

Now we notice that both the numerator and the denominator of $f'(x)$ are positive: $e^x > 0$ for all x and, therefore, $1 + e^x > 0$ also (so that $(1 + e^x)^2 > 0$ everywhere). This means that there are no critical points of $f(x)$, so that there are no local extrema. It also means that the function is increasing everywhere.

For the second derivative, the denominator is always positive, as is the factor e^x in the numerator. This means that if $f''(x) = 0$ then we'll need to have $1 - e^{2x} = 0$. You can verify that this occurs only at $x = 0$.

With this in mind, you can sample $f''(x)$ to the left and right of $x = 0$, and you'll find that $f''(x) < 0$ on the interval $(0, \infty)$ and $f''(x) > 0$ on the interval $(-\infty, 0)$. Hence f is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$. Since concavity changes at $x = 0$, this is an inflection point of the graph. $\square$