OPTIMIZATION PROBLEMS

1. Recap

Last class period we discussed absolute extrema, the extreme value theorem, and a procedure for optimizing a continuous function on a closed interval. We applied these techniques to solve certain kinds of optimization problems.

2. Optimization problems – on closed intervals

Today we're going to discuss how we apply the ideas in calculus (and in particular the topics we've discussed in the last 2 class periods) to optimization problems. These problems fall into two general categories: those which require optimization of a continuous function on a closed interval, and those that require optimization of a continuous function on an open interval. We talked about the former last class period, and today we'll see that most of the strategy carries over to the latter as well.

An optimization problem is essentially a word problem which asks you to maximize or minimize a certain quantity. In this regard, they are incredibly similar to the sorts of problems we've been doing the last few days. Nearly all optimization problems can be solved by employing a fixed strategy. It goes something like this

(1) Read and re-read the problem until you understand it. In particular, make sure you know the quantity you're being asked to maximize or minimize.
(2) Draw and label a picture which gives the relevant information.
(3) Write equations that describe
    (a) the quantity you're attempting to maximize/minimize in terms of other variables which appear in your drawing, and
    (b) the contraints that your variables must obey.
(4) Solve Equation (b) for one of the variables, and plug this result back into Equation (a).
(5) Determine the domain of your function. This is very important!
(6) Run the appropriate calculus machine.

For the last step, the 'machine' you run will depend on the kind of function you're optimizing. If it happens to be a continuous function on a closed interval, you follow the procedure we developed last class period for optimizing a continuous function on a closed interval. If it happens to be a continuous function on an open interval, the strategy will be different (we'll see how to solve these problems below).

Let's try one of these problems.

Example. A piece of wire 12m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a minimum and (b) a maximum? (Note: it is allowed to not cut the wire at all, but instead to use all of it to construct either a square or a triangle.)

Solution. After I've read the problems a few times to understand what it's asking, I like to draw a picture of what's going on. In the picture I'll label certain quantities so I can keep track of what's happening. For this problem, I'll draw my piece of wire cut into two pieces (leaving two pieces of length x and y) and what these pieces of wire look like after they've been fashioned into a circle and a triangle.

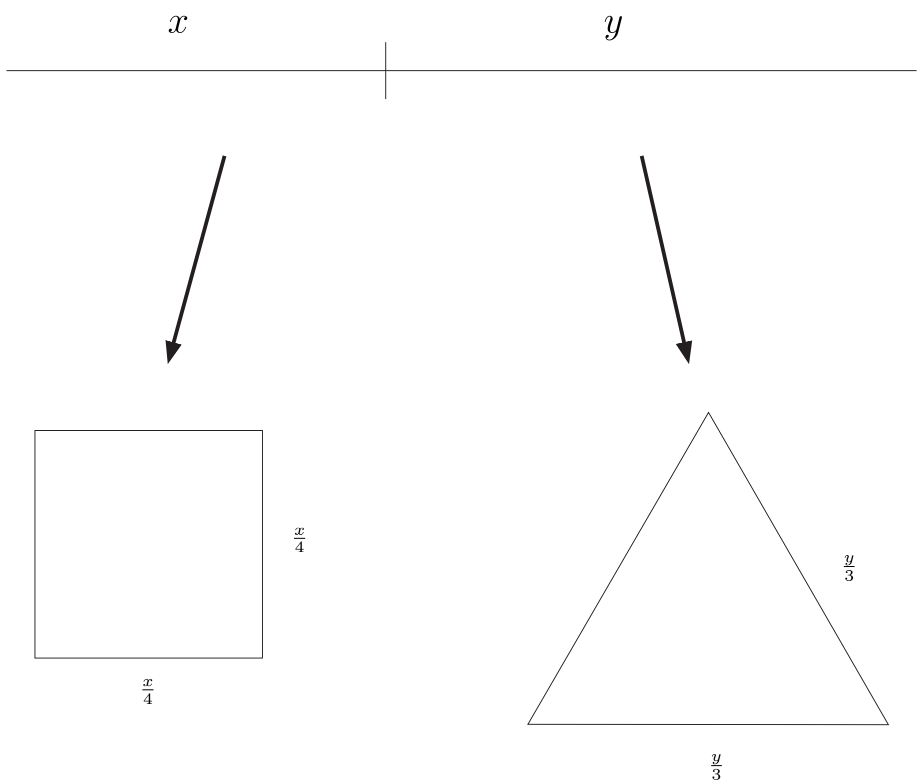

Figure 1. A schematic for our problem

We can see from the picture that the total area enclosed by both figures is

$$A = A_1 + A_2 = \left(\frac{x}{4}\right)^2 + \frac{\sqrt{3}}{4}\left(\frac{y}{3}\right)^2 = \frac{1}{16}x^2 + \frac{\sqrt{3}}{36}y^2$$

(the second term I know either from looking at the front of the book or by using the pythagorean theorem to solve for the height of the triangle). Perhaps a less obvious thing to write down is what I call a contraining equation; it's an equation that 'remembers' how the variables x and y relate to each other. In this case, the equation is given by

$$x + y = 12.$$

What I'd like to do is take my expression for area and optimize it. The problem is that currently it sits as a function of *two* variables instead of one. We don't have the technology yet to optimize a function of 2 variables. So what do we do? We'll take our constraining equation and use it to solve for y in terms of x: since $x+y = 12$, this means that $y = 12 - x$. Now I can substitute this value for y back into my expression for area to come up with an expression that makes area a function of one variable

$$A(x) = \frac{1}{16}x^2 + \frac{\sqrt{3}}{36}(12 - x)^2.$$

Now I have area as a function of the variable x. Notice that x can sit anywhere between 0 and 12, so we're now asking the following problem: find the absolute minimum (part (a)) and absolute maximum (part (b)) of the function $A(x)$ on the interval $[0, 12]$. How do we solve this problem? We use the three step technique we developed last class period.

First, we verify that $A(x)$ is a continuous function on the closed interval $[0, 12]$. Since it is a polynomial, it certainly is continuous.

Second, we find critical points for $A(x)$. Since

$$A'(x) = \frac{1}{8}x + \frac{\sqrt{3}}{18}(12 - x)(-1)$$

(I used the power rule and chain rule to compute this), we see first that the derivative is defined everywhere so that critical points are simply zeroes of the first derivative. When is $A'(x) = 0$? This happens if and only if $\frac{1}{8}x + \frac{\sqrt{3}}{18}(12 - x)(-1) = 0$, which is equivalent to

$$\frac{1}{8}x + \frac{\sqrt{3}}{18}x = \frac{12\sqrt{3}}{18} \Leftrightarrow x \approx 5.21.$$

(Of course, problems you'll encounter in quizzes or tests won't require a calculator to solve, so don't be too scared by the scary numbers).

Finally, we evaluate the function at these critcal points to find the maximum and minimum

x	$A(x)$
0	$\frac{12^2\sqrt{3}}{36} \approx 6.93$
5.21	$\frac{5.21^2}{16} + \frac{6.79^2\sqrt{3}}{36} \approx 3.92$
10	$\frac{12^2}{16} \approx 9$

From the table we see that area is maximized when $x = 12$ (that is, when all of the wire is used to make the square) and minimized when 5.21 meters is used to make the square (and so 6.79 meters is used to make a triangle). $\square$

3. Optimization problems – on open intervals

It often happens that you are asked to optimize a function on an open interval (or the whole real line) instead of a closed interval? How does one go about solving such a problem? Happily, much of the technique in optimizing functions on closed intervals will carry over to optimizing functions on open intervals. The main difference is that in finding absolute maxima or minima we will use a variation of the 1st derivative test instead of our techniques for finding absolute maxima and minima for continuous functions on closed intervals. This variation of the 1st derivative test is:

Theorem 3.1 (1st Derivative Test for absolute extrema). *Suppose $f(x)$ is a continuous function defined on (a, b) (where here $a = -\infty$ and $b = \infty$ are possible). If $f'(c) = 0$ or undefined and $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then f has an absolute minimum (in its domain) at c. If $f'(c) = 0$ or undefined and $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then f has an absolute maximum (in its domain) at c.*

Notice that this is very similar to the 1st derivative test for local extrema. The difference is that in this test we require information about the sign of the derivative for *all* $x < c$ and $x > c$; in the local test, we needed to only know this information for values of x near c.

Note.

- To use the 1st derivative test for absolute extrema directly, you should be in the situation where you have exactly one critical point in your domain. If you have more than one critical point in your domain, you probably have to do more work.
- If you're in the situation where you have exactly one critical point (say, c) in your domain, then to test the sign of the derivative for values $x < c$ it is enough to compute the sign of the derivative for one value of $x < c$. Similarly, to test the sign of the derivative for values $x > c$ it is enough to compute the sign of the derivative for one value $x > c$. We saw this when we did the first derivative test for local extrema, and said that it follows from the intermediate value theorem.

Let's use this test to work an example.

Example. Find two numbers whose sum is 23 and whose product is maximum.

Solution. Normally we would start a problem like this drawing a picture. For this problem, though, there's no real need. We see we're looking for numbers x and y such that $x + y = 23$ and so that their product P is maximized: $P = xy$. Notice in particular that x and y have no further restriction. In particular, x could be any real number. Now since $x + y = 23$, we have $y = 23 - x$. Hence we're trying to maximize

$$P(x) = x\,(23 - x) = 23x - x^2$$

on the domain $(-\infty, \infty)$. We'll use the 1st derivative test for absolute extrema.

Now $P'(x) = 23 - 2x$, so that $x = \frac{23}{2}$ is our only critical number. *Since we're in the situation where there is only one critical number*, to find the sign of the derivative to the left of $\frac{23}{2}$ and to the right of $\frac{23}{2}$, it is enough to find the sign of the derivative at some point to the left of $\frac{23}{2}$ and at some point to the right of $\frac{23}{2}$. Now 0 is to the left of $\frac{23}{2}$, and $P'(0) = 23 > 0$. Hence $P'(x) > 0$ for values $x < \frac{23}{2}$. Similarly, since $20 > \frac{23}{2}$ and $P'(20) = 23 - 40 < 0$, we see that $P'(x) < 0$ for values $x > \frac{23}{2}$. The 1st derivative test tells us $x = \frac{23}{2}$ is a local maximum of $P(x)$, as desired.

Hence, the two numbers we're looking for are $x = \frac{23}{2}$ and $y = 23 - \frac{23}{2} = \frac{23}{2}$. □