LINEARIZATION

1. Applications of the derivative

In class so far we've spent a lot of time talking about computing derivatives. We've seen that these computations can solve several problems for us, including

(1) providing a solution to the tangent problem;
(2) fueling a systematic approach to optimizing functions; and
(3) giving analytic tools for determining the geometry of the graph of a function.

Today in class we'll see how we can use the tangent line to the graph of $f(x)$ at a point a to approximate values of $f(x)$ at points near a.

2. Linearization

One of the overriding themes in calculus is that the tangent to the graph of a function $f(x)$ at a point $(a, f(a))$ is a decent approximation to $f(x)$ at points near a. One can see this geometrically by graphing the tangent at a point a and 'zooming in' to the point a. As you get closer to a, the tangent line begins to look just like the function.

This has a lot of great applications, but one of the most straightforward is that we can use the tangent line to $f(x)$ at a to approximation the values of $f(x)$ at points close to a. Since it's frequently hard to evaluate functions at random points without a calculator, this will give us a technique to approximate certain quantities with only a little calculus know-how.

There are plenty of functions that we can evaluate easily. For instance, evaluating polynomials is not very difficult, since they only involve operations like addition, subtraction, and multiplication. For the same reason, computing the value of rational functions is also relatively easy. But is it easy to compute the value of functions like $\sqrt{x}$ or $\log_2(x)$?

Functions like $\sqrt{x}$, $\log_2(x)$, and $\sin(x)$ are functions that we see all the time in calculus. Fortunately, there are some values of these functions we can compute easily. For instance, we all know that $\sqrt{4} = 2$ or that $\sin\left(\frac{\pi}{2}\right) = 1$. But could you easily compute $\sqrt{4.01}$? Without a calculator, this would be quite difficult to do. However, we can use the fact that the tangent line to f at a point a is a pretty good approximation of the function f 'near' a. Hence, if we can compute the equation of the tangent to f at a, we can use this to approximate the value of $f(a + \varepsilon)$ for small values of ε.

Example. Approximate the value of $\sqrt{4.1}$

Solution. Let $f(x) = \sqrt{x}$, so that we're attempting to approximate $f(4.1) = \sqrt{4.1}$. Notice that I know the value of $f(x)$ and $f'(x)$ at $x = 4$, and so I can solve for the equation of the line tangent to the graph of $f(x)$ at $x = 4$. Specifically, since $f(4) = 2$ and $f'(4) = \frac{1}{4}$, the equation of the line tangent to $f(x)$ at $x = 4$ is

$$y - 2 = \frac{1}{4}(x - 4) \quad \text{or} \quad y = \frac{1}{4}(x - 4) + 2.$$

Now we use linearization: this tangent line is supposed to be a good approximation to $f(x)$ near $x = 4$, and so we can estimage $\sqrt{4.1}$ by plugging 4.1 into the equation of the tangent line. This gives

$$\sqrt{4.1} \approx \frac{1}{4}(4.1 - 4) + 2 = 2\frac{1}{40}.$$

□

Example. Approximate the value of $e^{0.5}$.

Solution. Let $f(x) = e^x$, so that this time we're attempting to approximate $f(0.5) = e^{0.5}$. Notice that I know the value of $f(x)$ and $f'(x)$ at $x = 1$ (each is e since $f(x) = f'(x) = e^x$), and so I can solve for the equation of the line tangent to the graph of $f(x)$ at $x = 4$. The equation of the line tangent to $f(x)$ at $x = 1$ is

$$y - e = e(x - 1) \quad \text{or} \quad y = e(x - 1) + e.$$

Now we use linearization: this tangent line is supposed to be a good approximation to $f(x)$ near $x = 1$, and so we can estimage $e^{0.5}$ by plugging 0.5 into the equation of the tangent line. This gives

$$e^{0.5} \approx e(0.5 - 1) + e = \frac{e}{2} \approx 1.35.$$

Alternatively, we could write down the equation of the tangent line at $x = 0$. In this case $f(0) = f'(0) = 1$, and so the tangent line at 0 as equation

$$y = x + 1.$$

Using this line to approximate gives

$$e^{0.5} \approx 1.5.$$

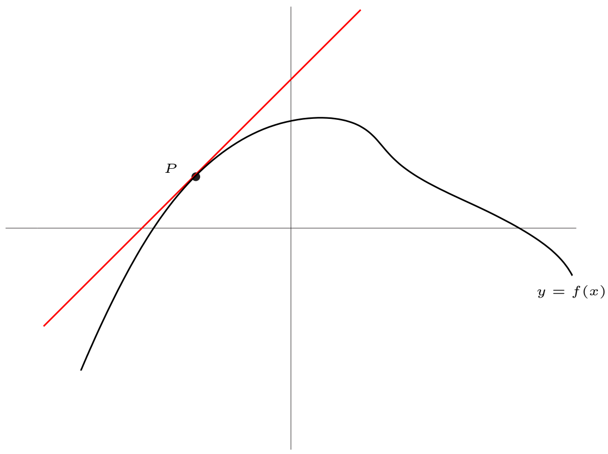
□

To emphasize: the point of linearization is that we can use the easy-to-evaluate tangent line to approximate the difficult-to-evaluate function provided we can find a point on the function whose tangent line we can write down.

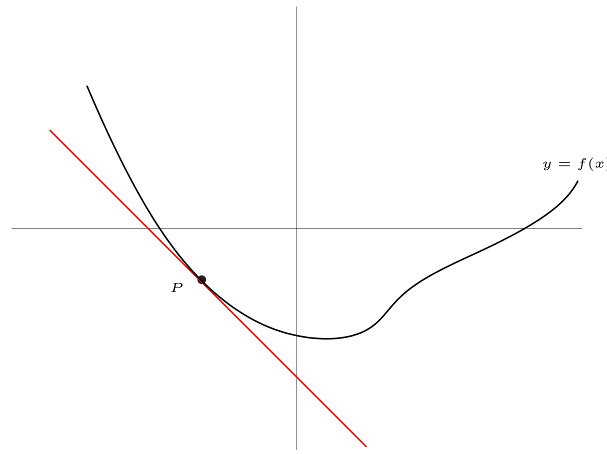
3. Precision

One will naturally ask how good these approximations are. There is an entire theory developed around stating specifically how trust-worthy these approximations are, but I won't worry about talking about that here. For this class, it's enough to notice that the tangent line to $f(x)$ at a point a provides a good approximation for $f(x)$ for points 'near' a. For example, if we used the tangent to $f(x) = e^x$ at $a = 0$ to approximate a^{100}, we should expect to get a really terrible approximation (since 100 is really far from 0).

A more tractable question is: are our approximations overestimates or underestimates? For this, notice that if $f(x)$ is concave down then the tangent to $f(x)$ sits above the graph of $f(x)$. Similarly, if $f(x)$ is concave up then the tangent line sits below the graph of $f(x)$.



Concave down means overestimate



Concave up means underestimate

This means that linearization (i.e., using the tangent line to approximate the values of $f(x)$) provides an overestimate in the case that $f(x)$ is concave down and an underestimate in the case that $f(x)$ is concave up.

Example. Was the approximation of $\sqrt{4.1}$ an overestimate or an underestimate?

Solution. We approximated $\sqrt{4.1}$ by looking at the tangent to $f(x) = \sqrt{x}$ at $x = 4$. Notice that $f''(x) = -\frac{1}{4}x^{-3/2}$, and that $f''(4) = -\frac{1}{4}(4)^{-3/2} < 0$. Hence $f(x)$ is concave down at 4, and from above we know that our approximation is an overestimate. $\square$

Example. Was the approximation of $e^{0.5}$ an overestimate or an underestimate?

Solution. We approximated $e^{0.5}$ by looking at the tangent to $f(x) = \sqrt{x}$ at $x = 1$ and $x =$. In either case we have $f''(x) > 0$, and hence $f(x)$ is concave up at 0 and 1. This means our approximations were underestimates (though the approximation 1.35 was more of an underestimage than the other approximation, 1.5). $\square$