COURSE OVERVIEW

1. Announcements

I've decided to cancel the last homework assignment (Homework 9). If you'd like to hand in the assignment, I'll be happy to grade it. If your score improves your overall homework average, I'll include it with your grades; if it hurts your overall homework average, I won't include it. You'll still get to drop your worst homework grade from the first 8 homework assignments.

Also, the review session on Sunday will be held at 4pm in room 383-N (as per usual). I will bring beverages and some 'light refreshments,' whatever that may means.

2. Course Overview

2.1. **Begin in the Beginning.** We began the class by discussing functions, objects which are the cornerstone of much of mathematics. We talked about basic properties of functions (the graph of a function, how to combine and uncombine functions) and proceeded to more advanced topics (inverses, graphs of inverses). We also built a library of basic functions: polynomials, rational functions, algebraic functions, trigonometric functions, exponential and logarithmic functions. These were the objects we wound up analyzing for the entire term.

2.2. **The Tangent Problem, Take 26.** After we had our feet solidly planted, we began to tackle the tangent problem. The tangent problem asks the following question. Given a function $f(x)$ and a point $P = (a, f(a))$ on the graph of $f(x)$, what is the equation of the line tangent to $f(x)$ at $(a, f(a))$? Of course to answer this

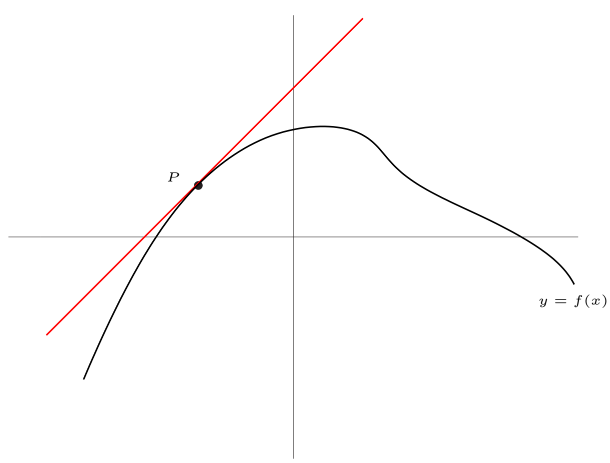

Figure 1. The function $f(x)$ and the tangent at $P = (a, f(a))$

question, we know from our investigation of lines that we need to know either 2 points on the tangent line or the slope of the tangent line together with a point on the line. But happily we already have a point on the graph (namely $(a, f(a))$), so we only need to know the slope of the tangent line. How will we do this?

The big idea is to consider secant lines. Specifically, plop a point $Q = (x, f(x))$ down on the graph of $f(x)$ and write the equation of the line through P and Q (you can do this since 2 points determine a line, just like we saw before). This is the so-called secant line through P and Q.

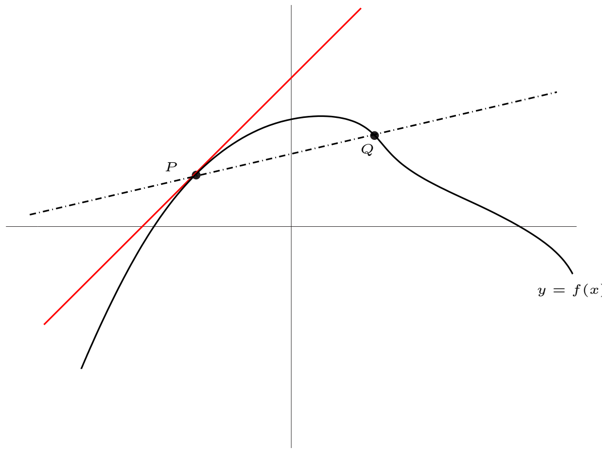

FIGURE 2. The secant line through P and Q

Now the magic: if we let Q approach the point P, the secant line will begin to approach the tangent line.

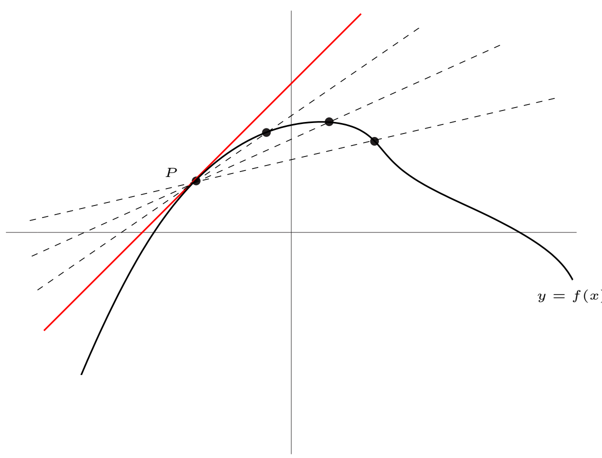

FIGURE 3. The secants approach the tangent as Q approaches P

So if we can study how the slope of the secant line changes as Q approaches P, we will have the value of the tangent line, as desired. In particular the slope of the secant line is $\dfrac{f(x) - f(a)}{x - a}$, and so the slope of the tangent will be

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

The problem is that we didn't yet have an understanding of limits were.

2.3. **Limits and $\frac{0}{0}$.** To remedy this situation, we began to investigate what limits were all about. We talked about a lot of basic properties of limits and saw that evaluating the limit of a function from its graph was a breeze. Unfortunately, the sorts of limits we were interested in are hard to graph. We were rescued by the fact that, in general, if we were asked to compute a hard limit we could do some algebraic manipulation and replace it by an easier to evaluate limit. Specifically, we could replace our problematic function with one that would be continuous at the limiting point, so that evaluating the limit would be equivalent to evaluating the given continuous function.

Example. Our prototype for this phenomenon was $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$. Both the numerator and denominator approach 0 as $x \to 1$, and so we know that we have to be clever in order to evaluate this limit. Noticing that for points $x \neq 1$ we have an equality $\dfrac{x^2-1}{x-1} = x+1$, we were able to conclude

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} x+1,$$

the second limit being easy to evaluate since $x+1$ is continuous.

2.4. **The Derivative.** Once we had a solid understanding of limits, we could do computations like the ones that motivated our investigation:

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

We gave this particular limit the name $f'(a)$ and called it the derivative of f at a. And we saw that it was good. And there was much rejoicing. We used this technique to compute slopes of tangents for all different kinds of functions and got quite good at solving the tangent problem. Indeed, we even wrote down a universal solution to the tangent problem

$$y - f(a) = f'(a)(x-a).$$

The bad news is that if we had a (fixed) function $f(x)$, to compute $f'(a)$ for lots of different values required lots of different limit evaluations. To remedy this problem we decided to keep the quantity 'a' variable, thus creating a function whose output was the slope of the tangent to $f(x)$ at the input value. We called this function the derivative of $f(x)$, and wrote it as both $f'(x)$ and $\dfrac{d}{dx}[f(x)]$. To compute it, we used a variant of our old rule for the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

2.5. **Tricks and Treats.** Using the definition of the derivative we were able to compute limits of lots of different functions, including things as mysterious as $\sin(x)$ and e^x (check out old course notes to see these computations). However, the limit calculations were getting tedious at this point because we were getting so wildly good at computing them. It was time that we learned some shortcuts for computing derivatives.

And learn shortcuts we did. We had tons. We saw how to express the derivative of a sum, product, quotient, and composition of functions in terms of the original functions (think: product, quotient, and chain rules). We also saw that $\dfrac{d}{dx}[x^n] = nx^{n-1}$, a very hand rule indeed. Using these rules together with the computations of the derivatives of e^x, $\sin(x)$ and $\cos(x)$ from earlier in the class, we could compute all kinds of wacky derivatives.

One missing link was $\dfrac{d}{dx}[\ln(x)]$. To solve this problem, we used a little sneaky thinking and the chain rule.

Example. Compute $\frac{d}{dx}[\ln(x)]$.

Solution. We're going to compute $\frac{d}{dx}[\ln(x)]$ in a tricky way.

First, we remember that $\ln(x)$ has the property that

$$e^{\ln(x)} = x.$$

Now let's compute the derivative of each side. On the right hand side we have $\frac{d}{dx}[x] = 1$. What about the left hand side? We compute this using the chain rule (with $f(x) = e^x$ and $g(x) = \ln(x)$). The chain rule says that $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$, and so we have

$$\frac{d}{dx}\left[e^{\ln(x)}\right] = e^{\ln(x)}\frac{d}{dx}[\ln(x)].$$

Setting these two derivatives equal to each other, we have $e^{\ln(x)}\frac{d}{dx}[\ln(x)] = 1$. Since $e^{\ln(x)} = x$, we can divide by x on both sides to obtain

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}.$$

□

This same trick allowed us to compute the derivatives of $\log_a(x)$, $\arcsin(x)$, $\arccos(x)$ and $\arctan(x)$. Indeed, these were the last pieces to the puzzle in being able to compute the derivative of even the most beastly functions. At this point in the class we could, if so inclined, compute something has gnarly as

$$\frac{d}{dx}\left[\frac{\arcsin(\sqrt{2^{\sin(x)+\ln(x)}})}{\tan(x^3+\sin(x))}\right].$$

This is truly an amazing amount of progress from where we started the class, so it's something you should feel really good about.

2.6. **What is it good for? The Tangent Problem again.** Once we were able to compute derivatives very well, we essentially mastered the art of solving the tangent problem. This makes us smarter than anyone that's 400 years old, and that's a lot of people to be smarter than.

2.7. **What is it good for? Optimization.** Another application of the derivative is in optimizing functions. We saw that finding critical points of a function allows us to determine when the function is locally maximum or minimum, based only on information about the sign of the derivative. We were able to jazz this technique up slightly to find absolute extrema of functions on both open and closed intervals.

Optimization is perhaps the most used applications of differential calculus and appears in almost all the sciences. For instance these techniques, in one guise or another, show up all over the place in economics and physics.

2.8. **What is it good for? Geometry.** Once we have information about the derivatives of a function we have a surprising amount of knowledge about its graph. Indeed, we developed a whole dictionary between properties of the graph of $f(x)$ and properties (really, sign information) of its derivatives $f'(x)$ and $f''(x)$. We didn't emphasize this in our class, but one could use this to give pretty refined sketches of the graph of $f(x)$.

2.9. **What is it good for? Linearization.** Another one of the big, big applications of differential calculus is linearization. The idea behind linearization (or linear approximation) is that the tangent to the graph $y = f(x)$ at the point $(a, f(a))$ is a good approximation to $f(x)$ for points 'close to' a. Indeed, once you zoom in around the point $(a, f(a))$, the graph of the function becomes almost indistinguishable from the graph of the tangent line. Since it's generally quite difficult to evaluate functions but always easy to evaluate the value of a line, this gives us a good way to approximate the value of a function $f(x)$ at points close to 'nice' points of $y = f(x)$ (ie, points where I know the equation of the tangent line).

2.10. **What is it good for? Finding zeros.** Another fantastic application of tangent lines is an algorithm called Newton's method which answers the age old question: for a function $f(x)$, what are the points that satisfy $f(x) = 0$. You know from your own mathematical experience that this is a question that comes up over and over again, but sadly there isn't generally a nice answer for finding the exact values of the zero of a function.

Here's a little history. For a function $f(x) = ax^2 + bx + c$, one can write down the zeros by the formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This has been known for at least 2000 years, but probably even longer. A more recent discovery (only a few hundred years old) is that if $f(x) = ax^3 + bx^2 + cx + d$ then the roots of $f(x)$ are given by an equation involving the coefficients a, b, c and d together with some basic operations (like adding, dividing, and extracting cube roots). A similar equation is known for degree 4 polynomials, so you might think that for any given polynomial there's some nice expression for its zeros in terms of the coefficients of the polynomial and easy operations (addition, multiplication, division, extracting roots).

A couple hundred years ago, some very clever people discovered that for a general polynomial $f(x)$ of degree at least 5 *there is no expression for the roots of $f(x)$ as an algebraic function of its coefficients*. This was totally amazing and still blows my mind!

So while this might make you sad, the good news is that there are a lot of ways to approximate the roots of a given function (i.e., solutions to $f(x) = 0$). One of these ways is called Newton's method, and it goes something like this. Plop a point $(x_0, f(x_0))$ down on your curve $y = f(x)$ and compute the tangent line ℓ_0 to the curve. Find the intersection of ℓ_0 with the line $y = 0$ and call this point x_1. Now the point $(x_1, f(x_1))$ has tangent line ℓ_1 which intersects the x-axis at a point which we'll call x_2.

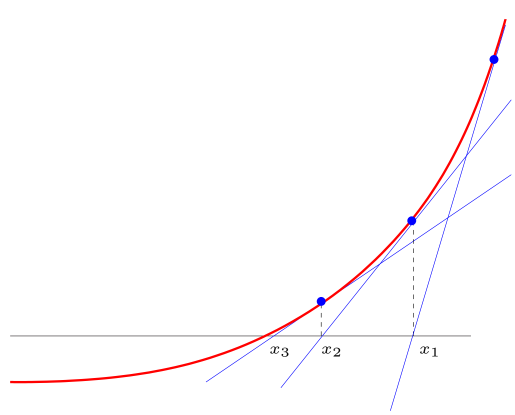

FIGURE 4. Newton's method at work

We keep going in this manner and get a sequence of numbers $x_0, x_1, x_2, \cdots$. (there's a better picture of this in your book on page 323). Except for some bad cases, these points approach a zero of your function $f(x)$ (as suggested by the graph we've drawn). What's even better, the technique to find the x_i is relatively easy (computing derivatives is not something that's too hard for a computer–or you–to do, and finding where a line intersects $y = 0$ is a breeze). What's better still is that the x_i's converge to a zero fast...even faster than they are 'supposed to.' This technique is really an amazing application of derivatives.