HOMEWORK 2 SOLUTIONS

(2.2.2) Explain what it means to say that

$$\lim_{x\to 1^-} f(x) = 3 \quad \text{and} \quad \lim_{x\to 1^+} f(x) = 7.$$

In this situation is it possible that $\lim_{x\to 1} f(x)$ exists? Explain.

Solution. To say $\lim_{x\to 1^-} f(x) = 3$ means that as inputs approach 1 from the left, outputs of the function $f(x)$ approach 3. To say $\lim_{x\to 1^+} f(x) = 7$ means that as inputs approach 1 from the right, outputs of the function $f(x)$ approach 7.

Since these directional limits disagree, we know that $\lim_{x\to 1} f(x)$ does not exist. □

(2.2.5) For the function g whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

Solution. (a) $\lim_{t\to 0^-} g(t) = -1$
(b) $\lim_{t\to 0^+} g(t) = -2$
(c) $\lim_{t\to 0} g(t)$ does not exist (since directional limits disagree)
(d) $\lim_{t\to 2^-} g(t) = 2$
(e) $\lim_{t\to 2^+} g(t) = 0$
(f) $\lim_{t\to 2} g(t)$ does not exist (since directional limits disagree)
(g) $g(2) = 1$
(h) $\lim_{t\to 4} g(t) = 3$

□

(2.2.6) Sketch the graph of the following function and use it to determine the values of a for which $\lim_{x\to a} f(x)$ exists:

$$f(x) = \begin{cases} 2 - x & , \text{ if } x < -1 \\ x & , \text{ if } -1 \leq x < 1 \\ (x-1)^2 & , \text{ if } x \geq 1. \end{cases}$$

Solution. We graph the function below. From this graph we see that $f(x)$ is not continous exactly at -1 and 1, so that the function is continuous elsewhere.

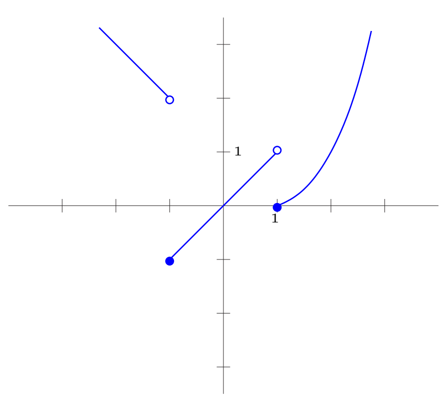

Figure 1. Solution to 2.2.6

$\square$

(2.2.9) Sketch the graph of an example of a function f that satsifies all of the given conditions: $\lim_{x\to 1^-} f(x) = 2, \lim_{x\to 1^+} f(x) = -2, f(1) = 2$.

Solution. There are lots of solutions. Below we give one possible solution.

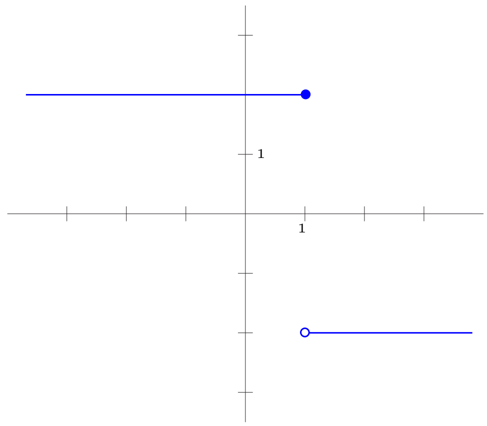

Figure 2. A possible solution to 2.2.9

$\square$

(2.2.10) Sketch the graph of an example of a function f that satsifies all of the given conditions: $\lim_{x\to -^-} f(x) = 1, \lim_{x\to 0^+} f(x) = -1, \lim_{x\to 2^-} f(x) = 0, \lim_{x\to 2^+} f(x) = 1, f(2) = 1, f(0)$ is undefined.

Solution. There are lots of solutions. Below we give one possible solution.

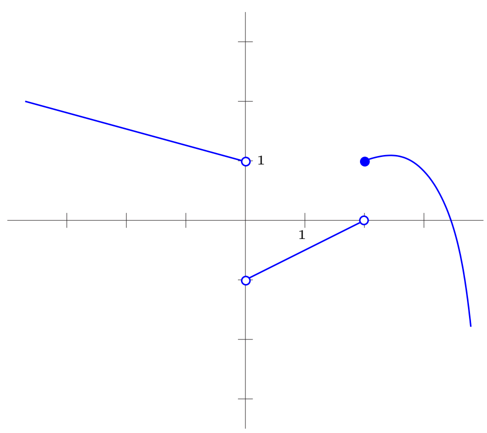

Figure 3. A possible solution to 2.2.10

$\square$

(2.2.13) Guess the value of the limit (if it exists)

$$\lim_{x\to 2}\frac{x^2-2x}{x^2-x-2}$$

by evaluating the function at the given numbers (correct to 6 decimal places).

Solution. With $f(x)=\dfrac{x^2-2x}{x^2-x-2}$, the problem asks us to evaluate $\lim_{x\to 2} f(x)$ by evaluating $f(x)$ at the given points. The chart below gives these outputs:

x	2.5	2.1	2.05	2.01	2.005	2.001
$f(x)$	0.714286	0.677419	0.672131	0.667774	0.667221	0.666778

x	1.9	1.95	1.99	1.995	1.999
$f(x)$	0.655172	0.661017	0.665552	0.666110	0.666556

We see from the chart that the values of $f(x)$ are approaching something in the neighborhood of 0.666 as $x\to 2$. Hence we guess that

$$\lim_{x\to 2}\frac{x^2-2x}{x^2-x-2}=\frac{2}{3}.$$

□

(2.3.2) The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.

Solution.

(a) Since f and g both have a limit at 2, we can apply our limit laws. We see then that $\lim_{x\to 2}[f(x)+g(x)]=\lim_{x\to 2} f(x)+\lim_{x\to 2} g(x)=2+0=2$.

(b) Since $\lim_{x\to 1} g(x)$ does not exist, we have to determine whether the given limit exists by comparing directional limits. Each diretional limit exists, so we can apply our rules:

$$\lim_{x\to 1^-}[f(x)+g(x)]=\lim_{x\to 1^-} f(x)+\lim_{x\to 1^-} g(x)=1+2=3$$
$$\lim_{x\to 1^+}[f(x)+g(x)]=\lim_{x\to 1^+} f(x)+\lim_{x\to 1^+} g(x)=1+1=2$$

Since the directional limits disagree, we know $\lim_{x\to 1}[f(x)+g(x)]$ does not exist.

(c) $\lim_{x\to 0}[f(x)g(x)]=\lim_{x\to 0} f(x)\cdot\lim_{x\to 0} g(x)$. But since $\lim_{x\to 0} f(x)=0$ and $\lim_{x\to 0} g(x)$ is some finite number, we have $\lim_{x\to 0} f(x)\cdot\lim_{x\to 0} g(x)=0$.

(d) To evaluate $\lim_{x\to -1}\dfrac{f(x)}{g(x)}$, we notice that the denominator has a limit approaching 0 while the numerator is approaching some non-zero number. Hence we know the limit in question does not exist.

(e) Since x^3 and f both have a limit at 2, we can apply our limit laws. We see that $\lim_{x\to 2}[x^3 f(x)]=\lim_{x\to 2} x^3\cdot\lim_{x\to 2} f(x)=8\cdot 2=16$.

(f) Since 3 and f are two function which have a limit at 1, so too does there sum. Hence we have $\lim_{x\to 1}\sqrt{3+f(x)}=\sqrt{\lim_{x\to 1}[3+f(x)]}=\sqrt{\lim_{x\to 1} 3+\lim_{x\to 1} f(x)}=\sqrt{3+1}=2$.

□

(2.3.8) What is wrong with the following equation?

$$\frac{x^2+x-6}{x-2}=x+3.$$

In view of this, explain why the equation

$$\lim_{x\to 2}\frac{x^2+x-6}{x-2}=\lim_{x\to 2}(x+3)$$

is correct

Solution. When $x \neq 2$, we have

$$\frac{x^2+x-6}{x-2} = \frac{(x-2)(x+3)}{x-2} = x+3$$

since in this case $\frac{x-2}{x-2} = 1$ (we need $x \neq 2$ so that $\frac{x-2}{x-2} \neq \frac{0}{0}$). Hence when $x \neq 2$, the equality holds. However, the equality doesn't hold at $x = 2$ since the left hand side isn't defined at 2, whereas the right hand side is.

Even though these functions disagree at 2, the equation

$$\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \lim_{x\to 2}(x+3)$$

is still correct because the two functions coincide away from $x = 2$. Indeed, in evaluating the limit we only consider what the function does near $x = 2$, and not what it does *at* 2. Since the two functions agree near 2, evaluating the limit of one is the same as evaluting the limit of the other. □

(2.3.9) Evaluate $\lim_{x\to 2} \dfrac{x^2+x-6}{x-2}$.

Solution. We saw above that

$$\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \lim_{x\to 2}(x+3),$$

and since $x+3$ is continuous at 2 we know that $\lim_{x\to 2} x+3 = 2+3 = 5$. Hence

$$\lim_{x\to 2} \frac{x^2+x-6}{x-2} = 5.$$

□

(2.3.15) Evaluate $\lim_{h\to 0} \dfrac{(4+h)^2-16}{h}$.

Solution. We begin by expanding the expression in the numerator. This will provide some cancellation so that, just as in the previous problem, we can reexpress the desired limit as the limit of a continuous function. Evaluating the limit will then be easy.

$$\lim_{h\to 0} \frac{(4_h)^2-16}{h} = \lim_{h\to 0} \frac{16+8h+h^2-16}{h} = \lim_{h\to 0} \frac{8h+h}{h} = \lim_{h\to 0} \frac{h(8+h)}{h} = \lim_{h\to 0} 8+h = 8.$$

□

(2.3.22) Evaluate $\lim_{t\to 0} \left(\dfrac{1}{t} - \dfrac{1}{t^2+t}\right)$.

Solution. In this problem I'll begin by expression the difference of fractions as one common fraction. Then I'll get some nice cancellation that will let me rewrite the limit in question as the limit of a continuous function. This will make evaluating the desired limit easy.

$$\lim_{t\to 0} \left(\frac{1}{t} - \frac{1}{t^2+t}\right) = \lim_{t\to 0} \frac{t^2+t-t}{t(t^2+t)} = \lim_{t\to 0} \frac{t^2}{t^2(t+1)} = \lim_{t\to 0} \frac{1}{t+1} = \frac{1}{0+1} = 1.$$

□

(2.4.1) Write an equation that expresses the fact that a function f is continuous at the number 4.

Solution. To say that f is continuous at 4 is, by definition, the same as saying

$$\lim_{x\to 4} f(x) = f(4).$$

□

(2.4.6) Sketch the graph of a function that has a jump discontinuity at $x = 2$ and a removable discontinuity at $x = 4$, but is continuous elsewhere.

Solution. There are lots of possible solutions. Here's one:

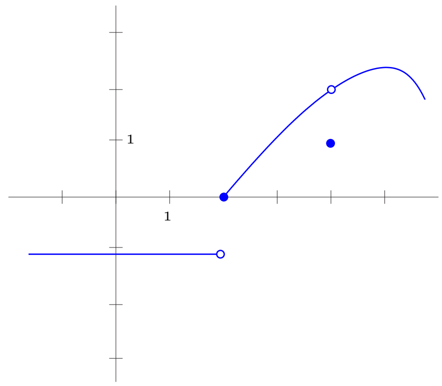

FIGURE 4. A possible solution to 2.4.6

□

(2.4.27) Evaluate $\lim_{x\to 1} e^{x^2-x}$.

Solution. Since $x^2 - x$ is a polynomial it is continuous, and we also know exponential functions are continuous. In class we said that the composition of continuous functions is continuous, and so we know e^{x^2-x} is continuous at 1. By the definition of continuity, we have

$$\lim_{x\to 1} e^{x^2-x} = e^{1^2-1} = e^0 = 1.$$

□